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DR. GURU PRASAD CHATTERJEE MEMORIAL LECTURE—1987

THE EVOLUTIONARY DYNAMICS OF QUANTITATIVE CHARACTERS*

P. NARAIN

Indian Agricultural Statistics Research Institute, New Delhi 110012

INTRODUCTION

The fact that evolution occurs is well recognised but what causes it to occur has been a matter of debate since the time Darwin[6] ascribed it to 'Natural Selection'. With continued natural selection for several generations, individuals possessing favourable characteristics have an advantage over those lacking them and as such they pass on such characteristics to their offspring for further modification and adaptation. Over the evolutionary time scale, this results in modified descendants of different ancestors living in geological times. However, for such a process to occur, it is necessary to have heritable variation and to understand factors governing it. Darwin unfortunately failed to discover the precise nature of such variation. It was only when Mendel's laws were re-discovered in 1900, that the precise nature of such variation, in the form of gene mutation as raw material for evolution to act upon, was properly understood. However, since the characteristics favoured during adaptive evolution are primarily quantitative in nature whereas mutation produce discontinuous variation, the compatibility between Darwinism and Mendelism was hotly debated for quite some time. Fisher[7] paved the way for a reconciliation between the two by showing how the observed correlation between relatives for such characteristics can be explained entirely in terms of the effects produced by Mendelian genes. Not only that, Fisher[9] further showed that the rate and direction of evolution are primarily determined by natural selection. According to his FundamentalT heorem of Natural Selection, the rate of increase in the average fitness of a population is proportional to additive genetic variance in fitness. Fisher[8,9] also investigated, for the first time, the problem of the maintenance of genetic variability in natural populations for quantitative characters and gave a model in which fitness was assumed to decrease in proportion to the squared deviation from the optimum. In this he initiated investigation on a problem which was not only followed by several workers subsequently but is still being vigorously pursued since the mechanism for the maintenance of such variability in natural populations is far from being well understood.

Subsequent to Fisher's work, Wright[30,31], Haldane[13], Robertson[26], Latter[22] and Bulmer[2,3] dealt with models based on multiple loci but each with only two segregating

(*Delivered on December 2 1987 at Indian Institute of Chemical Biology, Calcutta 700032)

alleles. Modern understanding of molecular genetics, however, indicates that a gene is subdivisible into a very large number of variable nucleotide sites so that a model involving segregation of infinitely many alleles at a given locus is more realistic than the early attempts. Crow and Kimura[5] introduced, for the first time, an infinite allele model which was later studied by Kimura[14] in detail. In this study, mutation was considered as producing multiple alleles with varying phenotypic effects in a continuous manner at each of the several loci involved in the inheritance of quantitative characters. Later on, Latter[23] adopted such a model in a discrete-time fashion. By extending this model to multiple loci, Lande[17–21] considered the effects of linkage and linkage disequilibrium. Fleming[10] studied extensively the entire equilibrium structure of the multi-allele, multi-locus case, both for discrete as well as continuous time situations.

The investigations of Latter[22] and of Bulmer[2,3] based on diallelic loci, led to the conclusion that the equilibrium genetic variance was independent of the magnitude of the phenotypic effects produced by mutation but depended on the total mutation rate, summed over the loci, and the intensity of selection. On the other hand, in the infinitely many allelle model of Kimura[14], the equilibrium genetic variance depended on the underlying biological parameters and the distribution of allelic effects was found to be Gaussian approximately.

The investigations of Turelli[27–29] were based on an alternative approximation for the continuum-of-alleles model of Crow and Kimura[5] on the empirically motivated assumption that the effects of new mutations at a locus are usually much greater than the existing genetic variance at the locus. This led to the prediction of the same equilibrium genetic variance as those for diallelic loci, predicted by Latter[22] and Bulmer[2,3]. Subsequently, Nagylaki[24], Gillespie[11], Gimelfarb[12] and Barton and Turelli[1] studied similar problems.

In these studies (with the exception of Latter[23], Lande[17], Gillespie[11], and Barton and Turelli[1]), the primary concern was to explain the magnitude of genetic variance within a population and little consideration was given to the genetic differentiation between populations or species. Moreover, none of these workers adopted a more realistic model of mutation involving discrete change of state. Chakraborty and Nei[4] however, developed a new multation model called "discrete allelic effect" to examine the extent of genetic variation of a quantitative trait within a population as well as the same between two populations during the process of their genetic differentiation. But their study considered the forces of mutation and random drift only and selection was ignored. Narain and Chakraborty[25] therefore, studied the evolutionary changes of genetic variance within and between populations for quantitative characters determined by a few loci with major effects by using the discrete-time, discrete allelic-state model with mutation and selection in an infinitely large population. The selection was of the optimum type so that we could examine the change in variance under two situations : (a) a population that evolves from monomorphism (at an optimum phenotype), and

(b) when an equilibrium population shifts to a new environment where the optimum phenotype is shifted by a few units of the phenotypic scale. The transient behaviour of the approach to equilibrium was studied in terms of changes in the means as well as the variances. The change in the interpopulational variance was examined when the equilibrium population splitted into two, where one of them moved to a new optimum environment.

In this paper, we critically review some of the models referred to above for studying the dynamics of quantitative characters, both in terms of the means as well as the variances, under mutation-selection balance. In addition, we also study the genetic differentiation between populations or species in respect of quantitative characters.

2. Alternative Approaches

The evolution of quantitative characters can be studied in either of the two ways. One way would be to define and study the underlying models at the level of phenotype avoiding any reference to gene frequencies. The Gaussian phenotypic analyses popularized by Lande[18,20] adopt this approach, on the assumption that genetic variances and covariances are known. Such an approach however, cannot predict the evolutionary dynamics of variance.

The other way could be to define and study a genetic model assuming that a complete genetic analysis of the traits is possible. In the latter case, one has to start from the simplest situation of a single locus with two alleles and build over it the more complex systems of infinitely many alleles at a locus, several loci, linkage and epistatic effects etc. The results obtained from the simpler situations of one or two loci give an insight to the problem, particularly for characters such as skin pigmentation in man which is believed to be controlled by a few loci say 5 to 6 with major effect. The rate of evolution, for such characters, is fairly rapid and modelling with few loci would be realistic. This latter approach would be mostly adopted in this paper.

3. Relationship Between Phenotypic and Genotypic Selection

The genetic properties of a given population are determined by gene and genotypic frequencies and need to be connected to quantitative differences noticed in a metric character at the phenotypic level. We consider a population of individuals with k genotypes $G_1, G_2, \ldots G_k$ with frequencies $f_1, f_2, \ldots, f_k$ and record the phenotypic measurements on each of the individual for the given character. Two individuals with the same genotype G_i may then differ slightly in their measurements due to random effects ascribed to environmental differences. To account for it, we take, for the genotype G_i, a phenotypic distribution $F(X_i/G_i)$ where X_i is the random variable giving measurement on the individuals with genotype G_i. The mean phenotype for this distribution is then

$$g_i = \int F_i f(X_i/G_i)\, dX_i. \qquad \ldots(1)$$

We thus have a sequence of k phenotypic distributions with means $g_1, g_2, \ldots, g_k$ corresponding to k genotypes but with a common environmental variance σ_E^2. The average and variance of the phenotypic values of the character in the population are then,

$$M = \sum_{i=1}^{k} f_i g_i$$

$$\sigma_P^2 = \sum_{i=1}^{k} f_i (g_i - \bar{g})^2 + \sigma_E^2 = \sigma_G^2 + \sigma_E^2 \qquad \ldots(2)$$

the latter expression indicating that σ_P^2 is the sum of the between-genotype variance (σ_G^2) and the average within-genotypic variance (σ_E^2) due entirely to environmental effects.

The change of the population mean resulting from the selection is brought about through the changes in the gene frequencies at the loci which influence the character under selection. But since the effects of the loci cannot be individually followed, the changes in the gene frequencies cannot, in practice, be ascertained unless we have some means of translating the phenotypic changes into genetic changes and vice-versa. The selection on the basis of the character X with a certain intensity induces selection among alleles at individual loci controlling the character. This is expressed in terms of a selection function $W(X)$ defined as relative selective value of an individual with measurement X. We can then use the phenotypic distributions together with $W(X)$ to determine the relative selective values W_i conferred on the corresponding genotypes. This is simply the average fitness of individuals with a given genotype. These fitness values can be used, along with gene frequencies, to determine the gene frequency in the next generation. This gene frequency information is then used to describe the mean and variance of the character in the next generation. The difference in the mean values between two successive generations gives the response to natural selection.

We follow the method of Kimura and Crow[15] to establish a general relation between the selection made at the overall phenotypic level and the consequent selection induced at the genotypic level at individual loci. Let X_{0P} be the optimum phenotypic value with maximum fitness. Consider a given locus with two alleles **A** and **a** segregating in the population with respective frequencies $(1 - q)$ and q. Assuming a random mating diploid population, let the average phenotypic values of **AA**, **Aa** and **aa** individuals be respectively X_{11}, X_{12} and X_{22}. Measuring the character in units of phenotyic standard dlviation σ_P, we denote the density (before selection) and fitness functions by $f(x)$ and $w(x)$ respestively. If we take X_{0P} as the origin, $x = (X - X_{0P})/\sigma_P$.

Let $m = (M - X_{OP})/\sigma_P$ and $a_{ij} = (X_{ij} - M)/\sigma_P$ where a_{ij} is the deviation of the average phenotypic value X_{ij} from the population mean in σ_P units. Let w_{ij} be the relative fitness of the genotype with value X_{ij}, then

$$w_{ij} = \int_{-\infty}^{\infty} w(x) f(x - a_{ij})\, dx. \qquad \ldots(3)$$

This is because the distribution of x in the sub-population of genotype with value X_{ij} is shifted by a_{ij} and the density function in this sub-population before selection is $f(x - a_{ij})$. This is a good approximation if the locus under consideration is contributing only a small portion of the total variance. Expanding $f(x - a_{ij})$ in Taylor series about $a_{ij} = 0$ and integrating, we get

$$w_{ij} = \beta_0 - a_{ij}\,\beta_1 + \frac{a_{ij}^2}{2}\,\beta_2 \qquad \ldots(4)$$

where we neglect third order terms in a_{ij}, treating it to be small, and

$$\beta_0 = \int_{-\infty}^{\infty} w(x) f(x)\, dx$$

$$\beta_1 = \int_{-\infty}^{\infty} w(x) f'(x)\, dx \qquad \ldots(5)$$

$$\beta_2 = \int_{-\infty}^{\infty} w(x) f''(x)\, dx.$$

If the effect of substituting **a** for **A** is α, the average genotypic values of **AA**, **Aa** and **aa** can be expressed as $a_{11} = -2q\alpha$, $a_{12} = (1 - 2q)\alpha$ and $a_{22} = 2(1 - q)\alpha$ respectively. Then

$$w_{11} = \beta_0 - a_{11}\,\beta_1 + \frac{a_{11}^2}{2}\,\beta_2 = \beta_0 + 2q\,\alpha\beta_1 + 2q^2\,\alpha^2\,\beta_2$$

$$w_{12} = \beta_0 - a_{12}\,\beta_1 + \frac{a_{12}^2}{2}\,\beta_2 = \beta_0 - (1 - 2q)\,\alpha\,\beta_1 + \left(\frac{1 - 2q}{2}\right)^2 \alpha^2\,\beta_2 \qquad \ldots(6)$$

$$w_{22} = \beta_0 - a_{22}\,\beta_1 + \frac{a_{22}^2}{2}\,\beta_2 = \beta_0 - 2(1 - q)\,\alpha\,\beta_1 + 2(1 - q)^2\,\alpha^2\,\beta_2.$$

The mean fitness of the whole population is

$$\bar{w} = (1 - q)^2\, w_{11} + 2q(1 - q)\, w_{12} + q^2\, w_{22}$$
$$= \beta_0 + q(1 - q)\,\alpha^2\,\beta_2. \qquad \ldots(7)$$

The gene frequency of **a** in the next generation is then

$$q' = (q^2\, w_{22} + q(1 - q)\, w_{12})/\bar{w} \qquad \ldots(8)$$

so that the change in gene frequency due to natural selection is

$$\begin{aligned}\Delta q &= q' - q \\ &= (q^2 w_{22} + q(1-q) w_{12} - q\bar{w})/\bar{w} \\ &= q(1-q)[-\alpha\beta_1 + \alpha^2 \beta_2 (\tfrac{1}{2} - q)]/\beta_0 = sq(1-q) \qquad \text{...(9)}\end{aligned}$$

neglecting terms involving α^3 and higher powers. This gives

$$s = -\alpha(\beta_1/\beta_0) + \alpha^2 \beta_2 (\tfrac{1}{2} - q)/\beta_0. \qquad \text{...(10)}$$

4. Mutation-Selection Balance at a Diallelic Locus

Stabilising selection reduces genetic variability, but in natural populations this is countered in each generation by fresh variability generated by mutation. We have therefore, to determine how much genetic variability is likely to be maintained by the balance between these two forces. Following Bulmer[3], we allow a mutation rate v from A_1 to A_2 as well as an equal mutation rate from A_2 to A_1. The expression for change in gene frequency given by (9) then becomes

$$\Delta q = q(1-q)[-\alpha(\beta_1/\beta_0) + \tfrac{1}{2}\alpha^2(\beta_2/\beta_0)(1-2q)] + v(1-2q). \qquad \text{...(11)}$$

Let the fitness function be

$$w(x) = \exp[-\tfrac{1}{2} x^2/\sigma_w^2] \qquad \text{...(12)}$$

and suppose the mean is zero when $q = \frac{1}{2}$ at all the loci controlling the trait. At equilibrium with $q = \hat{q}$, the q's must be symmetrical about 1/2.

Then

$$\begin{aligned}(\beta_1/\beta_0) &= 0 \\ (\beta_2/\beta_0) &\simeq -\ 1/(\sigma_P^2 + \sigma_w^2). \qquad \text{...(13)}\end{aligned}$$

This gives

$$(1 - 2\hat{q})[-\tfrac{1}{2}\alpha^2 q(1-q)/(\sigma_P^2 + \sigma_w^2) + v] = 0. \qquad \text{...(14)}$$

Thus $\hat{q}$ is either1/2 or it satisfies the quadratic equation

$$\alpha^2 \hat{q}(1 - \hat{q}) = 2v(\sigma_P^2 + \sigma_w^2). \qquad \text{...(15)}$$

Suppose now that there is an even number of loci ($2k$) and that (15) holds at all loci, half of the gene frequencies being at the smaller and half at the larger root. Since

$\sigma_g^2 = 2k\alpha^2 \hat{q}(1 - \hat{q})$, we have

$$\hat{\sigma}_g^2 = 4kv\,(\sigma_E^2 + \sigma_w^2)/(1 - 8kv)$$

$$\cong 4kv\,(\sigma_E^2 + \sigma_w^2)$$

$$= 2kv/s. \qquad \ldots(16)$$

As an illustration, suppose $\sigma_w^2 = 10\sigma_E^2$ (weak selection) and that $4kv = 0.01$ implying 250 loci each with a mutation rate of 10^{-5}, the expressed heritability under these rather favourable situations is only 0.10. The important point to note is that the equilibrium genetic variance is independent of the allelic effect. It depends only on the mutation rate and the intensity of selection.

5. Balance in an Infinitely Many Allele Model

The mutation model considered by Kimura[14] allowed for an infinitely many alleles at each locus. In this model, when a mutation occurs, it changes the value of the contribution of that allele by a small amount ξ from x to $(x + \xi)$ where ξ is a random variable with density function $f(\xi)$ which is symmetrical about zero with variance σ^2. A continuous time model is further assumed so that the fitness of an individual with value x is $-\frac{1}{2}x^2/\sigma_w^2$ where the optimum value is set at zero. This is equivalent to the usual model of stabilising selection with fitness $\exp\left(-\frac{1}{2}x^2/\sigma_w^2\right)$ in discrete time.

Let $p(x, t)$ be relative frequency of alleles with effect x at time t. The rate of change of $p(x, t)$ due to mutation is

$$\frac{\partial p(x, t)}{\partial t} = -vp(x, t) + v\int_{-\infty}^{\infty} p(x - \xi, t)\, f(\xi)\, d\xi$$

$$= v\sigma^2 \frac{\partial^2 p(x, t)}{\partial x^2} + O(\xi^2) \qquad \ldots(17)$$

where v is mutation rate per generation, $p(x - \xi, t)$ is expanded around $\xi = 0$ by Taylor series and

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} \xi\, f(\xi)\, d\xi = 0, \\ &\int_{-\infty}^{\infty} \xi^2 f(\xi)\, d\xi = \sigma^2, \\ &\int_{-\infty}^{\infty} \xi^n f(\xi)\, d\xi = 0,\ n \geqslant 3. \end{aligned}\right\} \qquad \ldots(18)$$

The rate of change of $p(x, t)$ due to selection is

$$\frac{\partial p(x, t)}{\partial t} = \tfrac{1}{2} p(x, t)\left[- x^2 + \int_{-\infty}^{\infty} x^2\, p(x, t)\, dx\right]/\sigma_w^2$$

$$= \tfrac{1}{2} p(x, t)\left[\sigma_g^2 - x^2\right]/\sigma_w^2 \qquad \ldots(19)$$

where

$$\int_{-\infty}^{\infty} x p(x, t)\, dx \to 0$$

$$\int_{-\infty}^{\infty} x^2 p(x, t)\, dx \to \hat{\sigma}_g^2$$

$$p(x, t) \to p(x)$$

as $t \to \infty$ in the steady state. At equilibrium, therefore, we have, $\partial p(x, t)/\partial t = 0$ and

$$\tfrac{1}{2} \nu\sigma^2 \frac{d^2 p(x)}{dx^2} - \tfrac{1}{2} p(x)(x^2 - \hat{\sigma}_g^2)/\sigma_w^2 = 0. \qquad \ldots(20)$$

This ordinary differential equation is satisfied when $p(x)$ is the density function of a normal distribution with mean zero and variance given by

$$\hat{\sigma}_g^2 = \sqrt{\nu\sigma^2\sigma_w^2}. \qquad \ldots(21)$$

Hence in the steady state, the balance between mutation of an infinitely many allele type and stabilising selection, in continuous time, produces a Gaussian distribution of allelic effects with zero mean and variance given by (21). The genetic value of the individual is therefore, distributed normally around zero mean with $2\hat{\sigma}_g^2$.

6. Balance with a Step-wise Discrete Mutation Model

We consider a quantitative character controlled by n loci and assume that at each locus there is an infinite number of possible allelic states. We assume that the phenotypic effect of the alleles are discrete as shown in Fig. 1.

In this figure, A_i represents an allele occupying state i (any integer number from $-\infty$ to ∞) and having an allelic effect of ia. We assume that all allelic effects are additive with no dominance and no epistasis and that once A_i mutates, it changes to allelic state A_{i+r} with probability.

$$\alpha_r = \alpha_{-r} = \binom{2m}{m-r} (\tfrac{1}{2})^{2m} \text{ for } 0 \leqslant r \leqslant m \qquad \ldots(22)$$

$$\alpha_r = 0, \text{ otherwise}$$

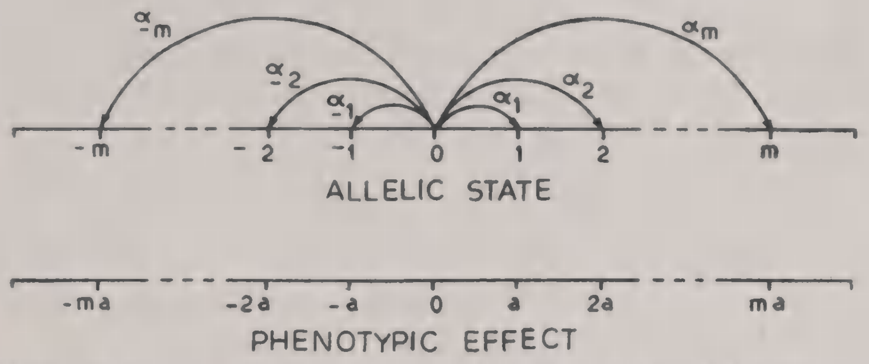

FIG. 1. Discrete allelic-state model used in this paper. In this model allele A_i mutates to A_{i+r} with probability α_r ($= \alpha_{-r}$), Allele A_i has a phenotypic effect of *ai*.

where m is the possible number of mutational steps. The distribution is a shifted Binomial with mean zero and variance $m/2$. If v denotes the mutation rate, the absolute probability of such a mutation would be $v\alpha_r$. Thus, an allele that mutates but has the same allelic effect as that of the original allele with probability $v\alpha_0$, so that in the conventional definition, the real mutation rate v' would be given by $v' = (1 - \alpha_0)\, v$. The per generation increment of the variance of allelic effect by mutation is then $vma^2/2$.

The selection operates on the total phenotypic value x and it is assumed to follow a normal distribution with mean μ and variance σ_p^2. With no genotype-environment interaction, the distribution of x, among individuals of type $A_i\, A_j$ with genotypic value $a\,(i + j)$ follows a normal distribution with mean $a\,(i + j)$ and the environmental variance σ_e^2 with density function

$$f\,(x/a\,(i+j)) = \frac{1}{\sigma_e\sqrt{2\pi}}\exp\left[-\frac{\{x - a\,(i+j)\}^2}{2\sigma_e^2}\right]. \qquad \ldots(23)$$

The fitness function for the character value x is assumed to be Gaussian type as

$$w\,(x) = w_{\max}.\ \exp\left[-\frac{(x - x_{\text{opt}})^2}{2\sigma_w^2}\right] \qquad \ldots(24)$$

where the character assumes the optimum fitness $w_{\max}$. at $x = x_{\text{opt}}$. and σ_w is the width of the function indicating the rate at which fitness declines with deviation of x from the optimum. Taking $w_{\max}. = 1$, the mean fitness of the individuals with genotype $A_i\, A_j$ would be

$$\begin{aligned} w_{ij} &= \int w\,(x)\, f\,(x/a\,(i+j))\, dx \\ &= \sigma_w \sqrt{2s}\exp\left[-s\,\{a\,(i+j) - x_{\text{opt}}.\}^2\right] \qquad \ldots(25) \end{aligned}$$

where $s = 1/2\,(\sigma_w^2 + \sigma_e^2)$ indicates the strength of the selection at the group level. A large σ_w means weak selection of the stabilising type.

If $x_i(t)$ denotes the frequency of allele $\mathbf{A}_i$ in generation t with allelic effect ai, an individual of genotype $\mathbf{A}_i\,\mathbf{A}_j$ will have a mean reproductive fitness $w_{ij} = \exp\,[-sa^2 (i+j)^2]$ and thus, the change in gene frequency of $\mathbf{A}_i$ from generation t to $t+1$ is given by

$$\bar{w}_A(t)\,x_i(t+1) = (1 - v + v\alpha_0)\sum_j x_i(t)\,x_j(t)\exp\,[-sa^2\,(i+j)^2]$$

$$+\, v\sum_{r=1}^{m}\alpha_r\,(\sum_j x_j(t)\,[x_{i+r}(t)\exp\{-sa^2(i+j+r)^2\}$$

$$+\, x_{i-r}(t)\exp\{-\,sa^2\,(i+j-r)^2\}]) \qquad ...(26)$$

where $\bar{w}_A(t)$ is the mean fitness of individuals at the locus in the tth generation so adjusted as to make $\sum_j x_j(t) = 1$.

In general, this recurrence relationship does not yield any explicit solution. However, for $m = 1$ it is possible to derive the equilibrium allele frequency profile by neglecting powers of v and s when the mean fitness $\bar{w}_A(t)$ is approximated as

$$\bar{w}_A(t) \simeq 1 - s\,\sigma^2_{g_A}(t) \qquad ...(27)$$

where $\sigma^2_{g_A}(t)$ is the total genotypic variance contributed by this locus at time t given by

$$\sigma^2_{g_A}(t) = a^2\sum_i\sum_j x_i(t)\,x_j(t)\,(i+j)^2. \qquad ...(28)$$

The recurrence relation reduces to

$$x_i(t) = \frac{v}{2}\,[x_{i+1}(t-1) + x_{i-1}(t-1)] + x_i(t-1)\,[1 - v - s$$

$$\times\,\{a^2 i^2 - \sigma^2_{g_A}(t-1)/2\}] \qquad ...(29)$$

giving

$$\Delta x_i(t) = -\,v\left[x_i(t) - \frac{x_{i+1}(t) + x_{i-1}(t)}{2}\right] + s\left[\frac{\sigma^2_{g_A}(t)}{2} - a^2 i^2\right]. \qquad ...(30)$$

When the population reaches equilibrium under the opposing pressures of mutation and selection, we have $\Delta x_i = 0$. This gives

$$\left.\begin{aligned} &\hat{x}_i = (1 - SG)\,\hat{x}_0 \\ &\hat{x}_{i+1} - 2\,[1 - S\,(G - i^2)]\hat{x}_i + \hat{x}_{i-1} = 0,\ i \geqslant 1 \\ &\hat{x}_i = \hat{x}_i,\ i \quad \text{non-zero integer.} \end{aligned}\right\} \qquad ...(31)$$

The frequent allele frequencies are obtained as

$$\left.\begin{aligned}\hat{x}_1 &= \hat{x}_0(1 - SG)\\ \hat{x}_2 &= \hat{x}_0[1 - 2SG + 2S(1 - G)(1 - SG)]\\ \hat{x}_3 &= \hat{x}_0[1 - 3SG + 6S(1 - G)(1 - SG)\\ &\quad + 2S\{3 - SG(6 - G) + 2S(1 - G)\\ &\quad \times (4 - G)(1 - SG)\}]\end{aligned}\right\} \qquad \ldots(32)$$

and

where $S = sa^2/v$ and $G = \frac{1}{2}\sum_i \sum_j \hat{x}_i \hat{x}_j (i + j)^2 = \sigma^2_{g_A}/2a^2$.

In the general case of m-step mutational changes, the moments of the allelic effects as well as those of genotypic effects can be obtained analytically under optimum selection. Denoting the kth moment of the distribution of allelic effects at a locus in the tth generation by $M_k(t) = \sum_{i=-\infty}^{\infty} a^k i^k x_i(t)$ and noting that for all i, $x_i(i) = x_{-i}(t)$ at each generation (since the optimum genotype is at origin), the recurrence relationship for the even order moments is given by

$$M_{2k}(t + 1) = [1 - v + v\binom{2m}{m}\left(\frac{1}{2}\right)^{2m}] M_{2k}(t) - M[M_{2k+2}(t)$$
$$+ s[M_2(t) M_{2k}(t) - M_{2k+2}(t)]$$
$$+ 2v\sum_{l=0}^{k}\binom{2k}{2l} M_{2l}(t) \sum_{r=1}^{m}\binom{2m}{m-r}\left(\frac{1}{2}\right)^{2m}(ar)^{2k-2l} \qquad \ldots(33)$$

so that the change of variance of allelic effects at generation t, $\Delta M_2(t)$ is given by

$$\Delta M_2(t) = \frac{mva^2}{2} + s[M_2^2(t) - M_4(t)]. \qquad \ldots(34)$$

At equilibrium, therefore, the fourth moment and the variance of allelic effects are related by

$$\hat{M}_4 = \hat{M}_2^2 + \frac{mva^2}{2s}. \qquad \ldots(35)$$

Now, if $\mu_r(t)$ denotes the rth order moment of the genotypic effects at the locus (i. e., $\mu_2(t) = \sum_i \sum_j a^r (i + j)^r x_i(t) x_j(t)$), the variance and the fourth moment of the genotypic effect at a locus are related with those of allelic effects by

$$\mu_2(t) = 2M_2(t) \text{ and } \mu_4(t) = 2M_4(t) + 6M_2^2(t).$$

We thus obtain the change of variance of genotypic values at a locus in generation t, $\Delta\mu_2(t)$, as

$$\Delta\mu_2(t) = mva^2 - s\,[\mu_4(t) - 2\mu_2^2(t)]. \tag{36}$$

If the equilibrium distribution of genotypic values is normal (i. e., $\hat{\mu}_4 = 3\hat{\mu}_2^2$), the genotypic variance at a locus in the equilibrium population is given by $\hat{\sigma}^2_{g_A} = \hat{\mu}_2 = \sqrt{mva^2/s} = \sqrt{\Delta\sigma^2_m/s}$ where $\Delta\sigma^2_m = mva^2$ is the effect of mutational change on the genotypic value at a locus. This result is identical to that of Kimura[14] even though his model assumes a continuous distribution of allelic effects.

Under the discrete allelic effect model, the departure from normality can be determined from the value of $\hat{\beta}_2 = \hat{\mu}_4/\hat{\mu}_2^2$ at equilibrium. We thus obtain

$$\hat{\beta}_2 = 2 + \Delta\sigma^2_m/s\hat{\sigma}^4_{g_A}$$

and hence the departure from normality as measured by $\gamma = \hat{\beta}_2 - 3$ is given by

$$\gamma = (\Delta\sigma^2_m - s\hat{\sigma}^4_{g_A})/s\sigma^4_{y_A}. \quad \ldots(37)$$

7. Genetic Differentiation between Populations or Species

Let us now consider the case where the populotion which is at equilibrium initially with mean phenotype at origin and variance σ_p^2 now shifts to a new environment where the optimum phenotype is $d\sigma_p$ units away from the mean, obviously, under the effect of mutation and selection the equilibrium status of genotypic distribution will be immediately disturbed and gradually the distribution will shift towards the new optimum. To analyze the nature and rate of change of variability we must again consider the recurrence relationship of gene frequency changes. In this new environment, the fitness, w_{ij}, of an individual of genotype $A_i\,A_j$ is given by

$$w_{ij} = \exp\,[-s\,\{a\,(i+j) - d\,\sigma_p\}^2].$$

We then get, following similar deivations,

$$\mu_1'(t+1) = \mu_1'(t) - s\,[\mu_3(t) + 2\mu_2(t)\{\mu_1'(t) - d\sigma_p\}] \quad \ldots(38)$$

where $\mu_r'(t)$ is the rth order moment (about origin) of the genotypic values in the tth generation. Clearly, at equilibrium we thus have $\mu_1' = d\sigma_p$, the optimum genotypic

value and $\mu_3 = 0$. The equilibrium distribution being symmetric, the change in genotypic variance is, similarly, given by

$$\Delta\sigma^2_{g_A}(t) = mva^2 + s\,[2\sigma^4_{g_A}(t) + \mu_3(t)\{(d\sigma_p) - 2\mu_1'(t)\} - \mu_4(t)] \quad ...(39)$$

where $\mu_3(t)$, $\mu_4(t)$ represent the third and fourth moments (about mean) of genotypic values at the tth generation. At equilibrium, since μ_3 is zero, we again have the same steady state genotypic variance given by

$$\sigma^2_{g_A} = \sqrt{\frac{1}{2}\left\{\hat{\mu}_4 - \frac{mva^2}{s}\right\}} \; . \quad ...(40)$$

Thus, even if the optimum is shifted by a certain s. d. away from the original mean, as long as the intensity of selection (s) remains the same, the genotypic variance eventually returns to its original value although the genotypic distribution becomes now centered around the new optimum genotypic value.

At the transitory stage however, it is difficult to assert analytically how the variance is altered. However, as we shall see in our numerical computations, at a transitory stage the genotypic variance first increases and eventually returns to its original equilibrium value.

It is apparent from the theoretical development given in the previous section that a quantitative study of genetic differentiation for metric traits between populations or species cannot be made analytically if optimal selection with stepwise mutation in an infinite population is envisaged. A computer was therefore used to compute numerically the various quantities of interest by resorting to exact recurrence relations already discussed in the previous sections. The mean and variability of the character both within as well as between populations were studied in the transient stage and at equilibrium. To start with, the initial population was considered as monomorphic at optimum and the behaviour of within population variance studied over time. After reaching the equilibrium, the optimum was shifted to a few units on the right and the transient behaviour of the mean and variability (within as well as between) was studied.

Since the mean is at the optimum and the optimum is set at zero, the population mean remains at zero unless there is a shift of the optimum genotype. However, the variance increases slowly from zero and attains, at equilibrium, a value determined solely by (v/s) as already obtained algebraically. This behaviour of within population variance as a function of time for different values of s between 0.004 to 0.040 for $m = 1$ and $v = 0.0005$ is presented in Table I.

For intense selection, the variances during the transient stage as well as at equilibrium are lower as otherwise expected. Mutation creates variability while selection eliminates it so that for intense selection, its role is dominant. Also, the approach to equilibrium

TABLE I

Within population variance at equilibrium when initial population is genetically homogeneous with only one type of individuals which is optimum phenotype for some selected values of selection coefficients (s). The mutation rate, v, is taken as 0.0005 and m = 1

s	$\hat{\sigma}^2_{g_A}$
0.004	1.17×10^{-1}
0.008	6.03×10^{-2}
0.010	4.85×10^{-2}
0.020	2.44×10^{-2}
0.040	1.22×10^{-2}

is quicker for more intense selection as revealed by results of computer simulations which are not presented here.

When we shift the optimum to four or six standard deviations away from the mean on the right and study the transient behaviour of the process as it approaches the same equilibrium, we notice some interesting results. In Table II, we present these results in terms of mean, variance, skewness and kurtosis of genotypic values in different generations for a quantitative character under centripetal selection when the new optimum is at six standard deviations away from the original optimum. We take $v = 0.001$, $s = 2v$, and $m = 5$.

TABLE II

Mean ($\bar{X}_t$), variance (V_t), skewness (γ_{1t}) and kurtosis (γ_{2t}) of genotypic values in different generations for a quantitative trait under centripetal selection ($v = 0.001$, $s = 2v$ and $m = 5$) with new optimum 6 standard deviations away from original optimum

	t					
	0	50	100	500	1000	∞
Mean($\bar{X}_t$)	0	0.378	0.930	3.443	3.811	4.000
Variance (V_t)	0.456	0.728	1.296	0.815	0.618	0.456
Skewnessγ_1 (γ_{1t})	0	0.997	0.684	0.119	—0.001	0
Kurtosis (γ_{2t})	2.741	1.886	0.433	0.764	1.563	2.741

N. B.: Kurtosis (γ_{2t}) is measured by $\gamma_{2t} = (\mu_{4t}/V_t^2)/^{-3}$.

The mean increases from zero to four at equilibrium but the increase is more rapid for a higher value of m. The variance on the other hand increases, attains a transitory maximum and decreases back to the original steady state value which is related to the fourth moment of the genotypic values as given by (40). When s is very large compared to v, more than one transitory maxima are produced. In Fig. 2, we present such a

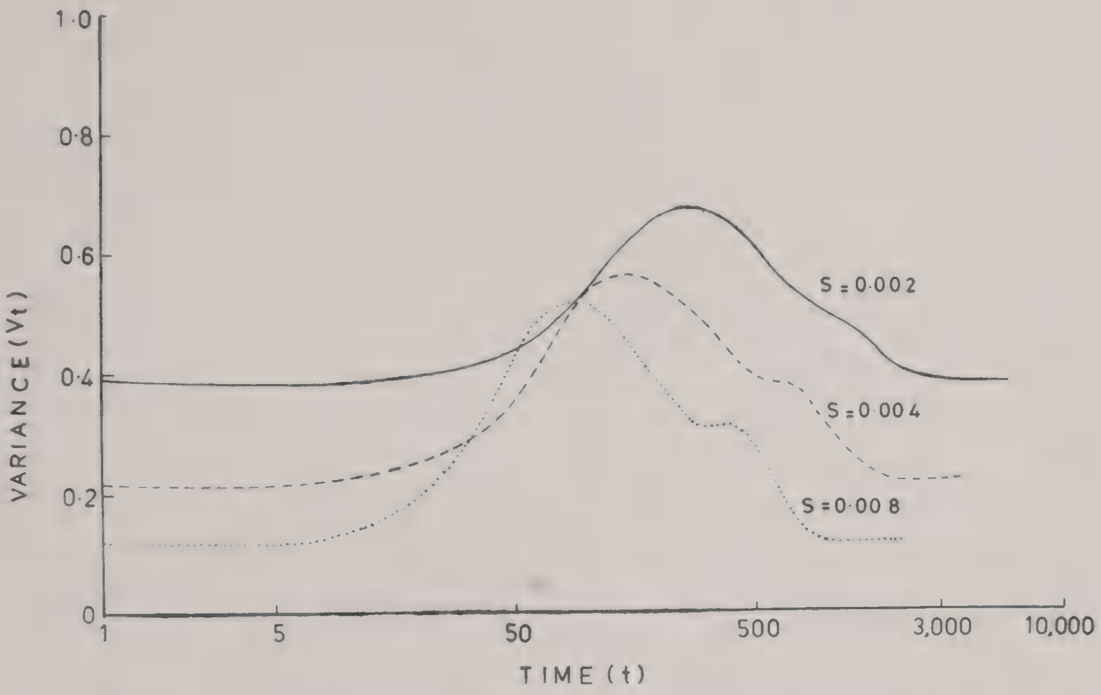

FIG. 2. Intra-population variance (V_t) under the joint effects of mutation and centripetal selection as a function of time t, with $m = 1$, $v = 0.001$, $s = 0.002$, 0.004 and 0.008 when the new optimum is at 6 standard deviations away from the original optimum at the origin. Initial population is at steady-state.

behaviour of intra-population variance for $s = 0.002, 0.004$ and 0.008 when $v = 0.001$, $m = 1$ and when the new optimum is at six standard deviations away from the original optimum at the origin.

For a higher value of m, the variance attains a considerably higher peak as well as somewhat earlier than when $m = 1$. The most interesting feature is regarding the skewness of the distribution of genotypic values. Initially, this distribution is symmetrical but as we advance in time, its symmetry is disturbed. It gets skewed initially and then slowly the skewness decreases, changes sign and finally the distribution becomes again symmetrical at equilibrium. The kurtosis of the distribution also behaves in a similar fashion. Starting from a value very near to three initially, it declines to a value less than half but increases thereafter and restores the initial value at the equilibrium. Compared to $m = 1$, the distribution for $m = 5$ becomes more leptokurtic. A typical allelic distribution corresponding to $v = 0.001$, $s = 2v$ and $m = 5$, depicting these features, is shown in Fig. 3.

Of special importance in these studies is the genetic differentiation between populations built up over a period of time when in one population the same optimum holds but in the other it has shifted a certain distance away from the mean. Chakraborty and Nei[4] used the ratio of between population (B_t) to within population (V_t) variability as an index for determining the evolutionary forces under which the character changes

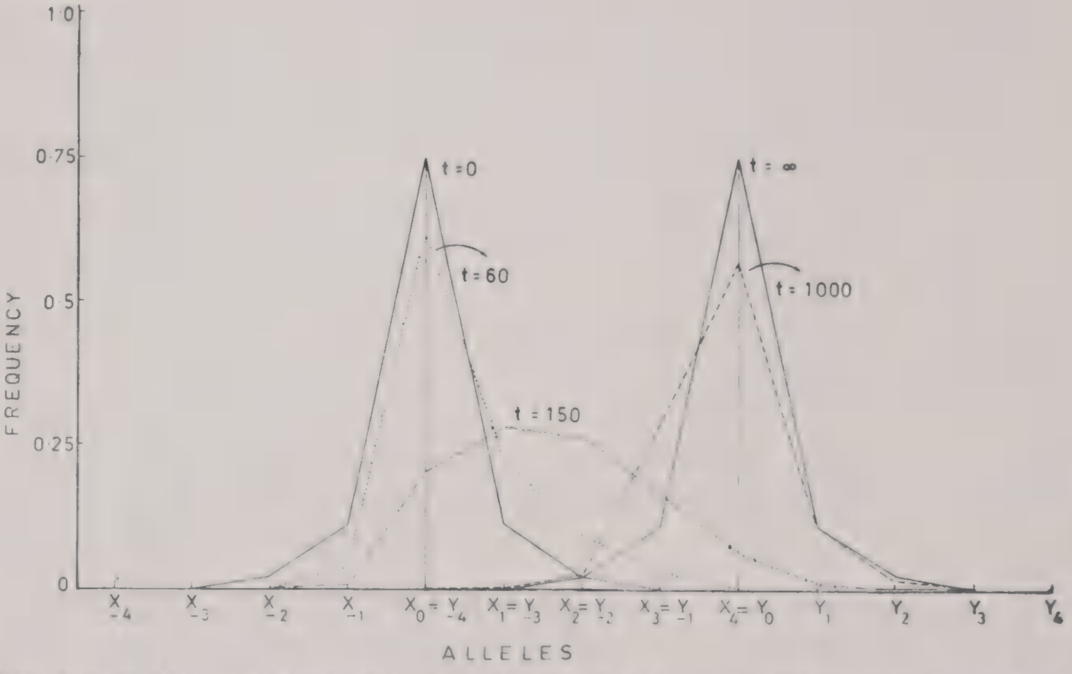

FIG. 3. Allele frequency distribution under joint effects of mutation and centripetal selection at different generations (t = 0, 60, 150, 1000 and ∞) with $\nu = 0.001$, $s = 2\nu$ and $m = 5$ when the new optimum is 6 standard deviations away from the original optimum at the origin. Initial population is at steady-state.

over time. Their studies involving drift only showed that this ratio (B_t/V_t) increases linearly with time. With centripetal selection in an infinitely large population we find that this behaviour changes considerably and it is no longer a monotone function of time.

We have already discussed the behaviour of V_t which increases slowly from initial equilibrium value, attains a maximum and then decreases back to the same value at equilibrium. But when we consider between population variability, B_t, it is found that it increases slowly initially and then almost linearly until it approaches a plateau at equilibrium. The ratio (B_t/V_t) almost mimics the behaviour of B_t at least in the initial stages but it attains a much higher value. This is obvious because V_t decreases while B_t increases as equilibrium is reached. In the initial transient stage, however, B_t and the ratio are almost the same because V_t has been increasing and reaching a maximum. After this stage, at which maximum V_t occurs, the quantities B_t and (B_t/V_t) diverge apart, increasing with time by different magnitudes. To illustrate the qualitative nature of the changes in V_t, B_t and (B_t/V_t), under joint effects of mutation and centripetal selection, Fig. 4 presents the numerical results for $\nu = 0.001$, $s = 2\nu$ and $m = 5$ when the new optimum is taken to be approximately six standard deviations away from the optimum in the other population.

It is seen from this figure that the rate of approach to equilibrium variance within population (0.456) is faster as compared to that of the mean genotypic value (4.0) since

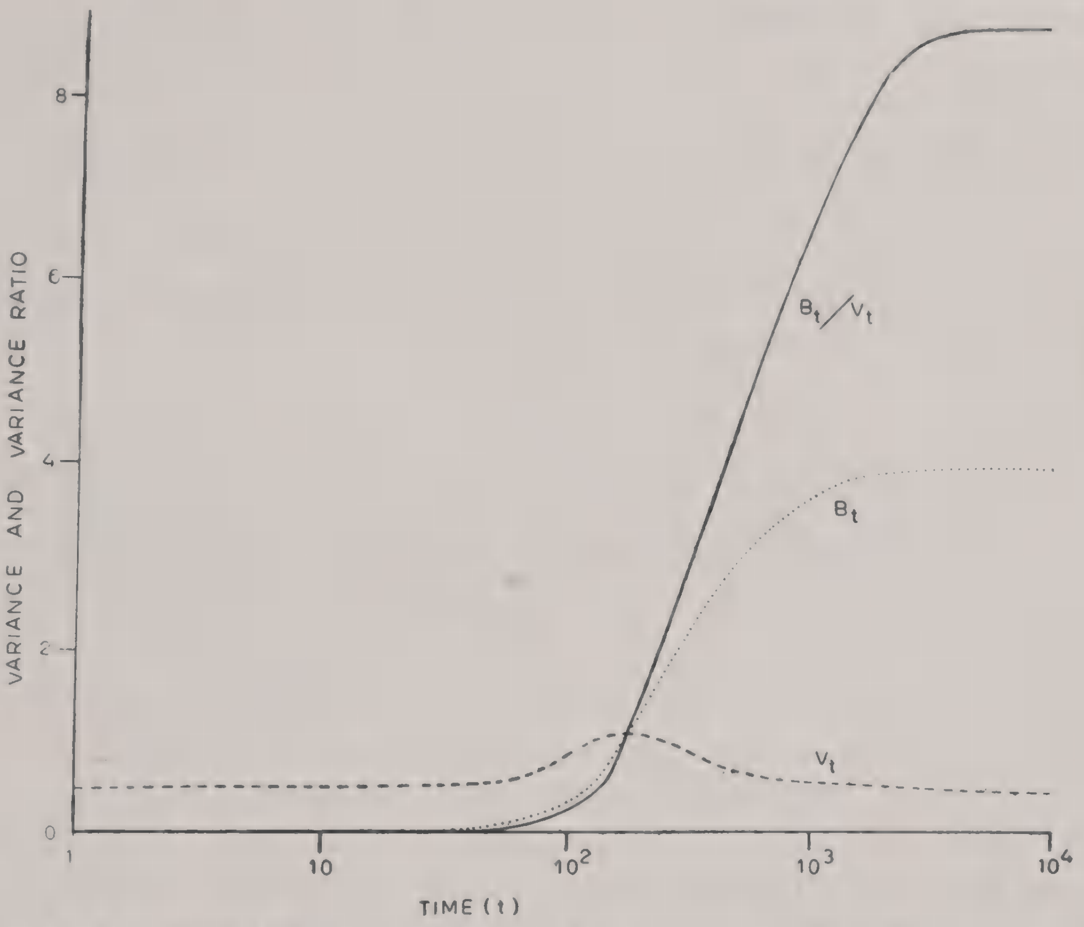

FIG. 4. Intra- (V_t) and inter-populational (B_t) variance components under the joint effects of mutation and centripetal selection as a function of time of divergence of two populations one of which has optimum genotye six standard deviations away from the other (which remains at a steady state by mutation-selection balance). Time (t) is measured in units of generation. Initial population is at steady state. The parameter values are $v = 0.001$, $s = 2v$ and $m = 5$.

the between-population variance (B_t) attains its equilibrium value (4.0) at a later time than V_t. The variance ratio (B_t/V_t) reaches a steady state value (8.80) since the process of genetic differentiation stops once the diverging population reaches its steady-state genotypic distribution around its new optimum value. We thus see that (B_t/V_t) asymptotes and does not therefore increase linearly with time as in Chakraborty and Nei[4] under mutation-drift balance. The behaviour of (B_t/V_t) as a function of time of divergence can therefore be taken as a criterion for determining whether selective forces are operating or not.

8. DISCUSSION

Kimura[14] showed, for infinitely large populations, and assuming a continuous time process that the distribution of allelic effects tends to be normal at equilibrium between selection and mutational forces and that the mean and variance of the equilibrium

distribution are determined by the amounts of increase in mean and variance of the genotypic value per gene per generation as well as by the intensity of fitness function. The discrete allelic-state model with the assumption of a discrete-time process, considered in this paper, has revealed behaviour similar to those of Kimura[14] as it should, since binomial distribution of allelic effects should tend to normal distribution as we go from discrete to continuous case. On the other hand, the diallelic model of Latter[22], reanalysed by Bulmer[2,3] who used different approximations but arrived at the same conclusions as those of Latter[22], indicated different results. Their predictions for equilibrium genetic variance differ qualitatively from those of Kimura-Lande-Fleming-Narain & Chakraborty.

Based on "House-of-cards" approximation of Kingman[16], Turelli[27] presented a new asymptotic analysis of Kimura's model to show that the qualitatively different predictions about equilibrium genetic variance are not due to the number of alleles assumed per locus. Instead, such different results are attributable to assumptions concerning the relative magnitudes of per locus mutation rates, the phenotypic effects of mutation and intensity of selection. He then analysed a model with tri-allelic loci which allows among-locus variation in mutation rates and allelic effects. At the single locus level, such a model is a particular case of the discrete-time, discrete allelic-state mutation model analysed in the present paper and leads to the same conclusions as in Turelli[27] in so far as the equilibrium genetic variance is concerned.

As already mentioned in the Introduction, very few studies on the problem of genetic differentiation between population or species have appeared, particularly for the situation when one of the daughter populations has a shifted optimum. Such cases have biological relevance as for instance in skin pigmentation for a small group of Caucasian race with fair skin who moved out of Central Asia around 3000 years ago and settled in southern parts of America with plenty of sunlight. The optimum phenotype for skin pigmentation must therefore, have shifted by several standard deviations away from the original optimum. This introduces differentiation between populations and it is of interest to study the transient properties of such a process. This has been done in this paper by studying the distribution of allelic frequency as a function of time. Significant changes in statistical properties of the distribution such as mean, variance, skewness and kurtosis have been noticed. In particular, algebraic expressions for changes in mean and variance reveal interesting results. When we consider optimum at d standard deviations units away from the origin, the mean change in the genotypic value of the character at the t-th generation, as given by eqn. (38), depends on s, d, σ_p, $\mu_3(t-1)$, $\mu_2(t-1)$ and $\mu_1'(t-1)$. It does not depend on the mutation rate, v. But change in the genotypic variance per generation, as given by eqn. (39), clearly indicates that it is affected by the mutational component. Initially, the population is in equilibrium and symmetrical. If we further assume that it has normal kurtosis, we have $\mu_1'(0) = 0$, $\mu_3(0) = 0$, $\mu_4(0) = 3\mu_2^2(0)$. If we also disregard the increase in varia-

tion due to mutation, the expression (39) reduces to the result given in Latter[23]. The genotypic variance is reduced and the reduction depends on the intensity of selection and the heritability.

The transient behaviour of the mean noticed in this investigation can be useful in giving some idea about the divergence time; at least the time by which the mean gets to half of the total change in the mean, i. e., 1/2 ($d\sigma_p$), which is known. The computer results show that intense selection speeds up the divergence time but by increasing the number of mutational steps, this time is shortened, though not very appreciably. Thus, for $v = 0.001$, $s = 0.008$ and $m = 1$, this time is of the order of 120 generations. With $s = 0.002$ and $m = 1$ and 5, the times are respectively 300 and 170 generations.

The main focus of this study is on the genetic differentation between populations which gets built up over time when the population, after reaching equilibrium by mutation-selection balance, splits up into two in one of which the same optimum holds but in the other it shifts a few standard deviations away from the mean. The ratio between versus within population variance (B_t/V_t) as a function of time then provides with a possible mean of ascertaining the role of adaptive changes under which the character changes over time. With selection, this ratio necessarily changes non-linearly with time.

Summary

Studies on the maintenance of genetic variability for a quantitative trait due to a balance between stabilising selection and mutation in natural populations have been reviewed. Several models, both in terms of the dynamics of the means as well as the variances have been discussed. In particular, using a new and more general genetic model called the discrete-allelic state model and assuming discrete-time process, the evolutionary changes of genetic variation of quantitative characters, controlled by a few loci, within and between populations during the process of genetic differentiation of populations or species, are studied under the effects of mutation and centripetal selection in infinitely large populations. While in a finite population and ignoring selection, the rate of change of additive genetic variance depends on mutation and effective population size, traits under optimal selection in infinitely large populations go through the dynamics of a rather complicated form depending on the relative intensities of selection and mutation. When a population, which has reached steady-state by mutation-selection balance, splits into two, in one of which the same optimum genotype holds but in the other the optimum shifts a few standard deviations away from the original optimum, the corresponding daughter population starts differentiating from its sister population by favouring certain class of mutant alleles and discarding others which were originally favoured. During this process of turn over of genes, both the intra- and inter-populational variances undergo a complicated change, and the ratio of the latter to the former is a non-linear function of time of divergence. This pattern is qualitatively very different from the case when selection is absent. The intra-population distribution of genotypic values, during this transition, is shown to deviate considerably from normality.

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ON SELECTING k BALLS FROM AN n-LINE WITHOUT UNIT SEPARATION

B. S. EL-DESOUKY

Statistical Laboratory, Department of Mathematics, University of Manchester Manchester, M13 9PL

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Let $l(n, k)$ denote the number of ways of selecting k balls from n balls arranged in a line (called an n-line) with no two adjacent balls, from the k selected balls, being unit separation. It is shown that $l(n, k)$ satisfies the recurrence relation

$$l(n, k) = l(n-1, k) + l(n-1, k-1) - l(n-2, k-1) + l(n-3, k-1)$$

and an explicit form for $l(n, k)$ is obtained.

1. INTRODUCTION

Kaplansky[1] proved that,

$$f(n, k) = \begin{cases} \binom{n+1-k}{k} & 0 \leqslant k \leqslant \frac{n+1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

where $f(n, k)$ denotes the number of ways of selecting k balls from n balls arranged in a line without any two selected balls being consecutive, (see also Riordan[3], Ryser[5]).

Recently Konvalina[2] proved that

$$g(n, k) = \begin{cases} \sum_{i=0}^{[k/2]} \binom{n+1-k-2i}{k-2i} & \text{if } n \geqslant 2(k-1) \\ 0 & \text{if } n < 2(k-1) \end{cases}$$

where $[k/2]$ is the greatest integer less than or equal to $k/2$ and $g(n, k)$ denotes the number of ways of selecting k balls from n balls arranged in a line (called an n-line) without any two selected balls being uni-separate (i. e. being separated by exactly one ball which can be either selected or not).

A related problem is to determine the number of ways of selecting k balls from an n-line with no two adjacent balls, from the k selected balls, being uni-separate. Let $l(n, k)$ denote the number of ways of selecting k balls from an n-line with no two adjacent balls, from the k selected balls, being uni-separate.

In this paper we derive a recurrence relation for $l(n, k)$ and then give a method similar to that used by Riordan[4] to determine $l(n, k)$.

2. The Main Result

Lemma 2.1—Let $l(n, 0) = 1$, then $l(n, k)$ satisfies the recurrence relation

$$l(n,k) = l(n-1,k) + l(n-1, k-1) - l(n-2,k-1) + l(n-3, k-1) \qquad \ldots(2.1)$$

with the boundary conditions $l(n, 1) = n$ and $l(1, k) = 0$ if $k > 1$.

Proof: To prove the recurrence relation (2.1), let $l(n, k)$ be defined as before. Then, the corresponding selections either contain the first ball or they do not. If they do not, then they are enumerated by $l(n-1, k)$. If they do, then the selections either contain the second ball or they do not. If they do not, then they cannot contain the third ball and, hence, are enumerated by $l(n-3, k-1)$. If the selections contain the first and second balls, then the selections either contain the third ball or they do not. If they do not they cannot contain the fourth and, hence are enumerated by $l(n-4, k-2)$. If the selections contain the first, the second and the third balls, then the selections either contain the fourth or they do not, and so on. Hence, $l(n, k)$ satisfies the recurrence

$$\begin{aligned} l(n, k) &= l(n-1, k) + l(n-3, k-1) + l(n-4, k-2) + \ldots \\ &= l(n-1, k) + l(n-3, k-1) + L(n, k) \end{aligned}$$

where

$$L(n, k) = \sum_{i=2}^{k} l(n-2-i, k-i).$$

Then

$$\begin{aligned} l(n-1, k-1) &= l(n-2, k-1) + l(n-4, k-2) \\ &\quad + l(n-5, k-3) + \ldots \\ &= l(n-2, k-1) + L(n, k) \end{aligned}$$

thus the recurrence relation (2.1) is obtained.

Theorem 2.1—

$$l(n,k) = \sum_{i=0}^{\lambda} \binom{k-1}{i} \binom{n-k+1-i}{i+1} \text{ if } k \leqslant n$$

where

$$\lambda = \min\left(k-1, \left[\frac{n-k}{2}\right]\right)$$

and $l\,(n, k) = 0$ otherwise.

PROOF : First we find $l\,(n, k)$ for $k = n, n - 1$ and $n - 2$. It is clear that $l\,(n,n) = 1$, from the boundary conditions and using the recurrence (2.1), we obtain

$$\begin{aligned} l\,(n, n-1) &= l\,(n-1, n-1) + l\,(n-1, n-2) - l\,(n-2, n-2) \\ &\quad + l\,(n-3, n-2) \\ &= l\,(n-1, n-2) = \ldots = l\,(2, 1) = 2 \\ l\,(n, n-2) &= l\,(n-1, n-2) + l\,(n-1, n-3) - l\,(n-2, n-3) \\ &\quad + l\,(n-3, n-3) \\ &= l\,(n-1, n-3) + 1 \\ &= l\,(3, 1) + n - 3 \\ &= 3 + n - 3 = n. \end{aligned}$$

It remains to find the explicit form for $l\,(n, k)$ if $k \leqslant n - 3$. From the boundary conditions $l\,(n, 0) = 1$, $l\,(n, 1) = n$, and using the recurrence (2.1) we have

$$\begin{aligned} l\,(n, 2) &= l\,(n-1, 2) + l\,(n-1, 1) - l\,(n-2, 1) + l\,(n-3, 1) \\ &= l\,(n-1, 2) + (n-2) \\ &= l\,(2, 2) + (n-2) + (n-3) + \ldots + 1 \\ &= \frac{(n-1)(n-2)}{2} + 1 = \frac{(n-2)(n-3)}{2} + (n-1) \\ &= \binom{n-1}{1} + \binom{n-2}{2}. \end{aligned}$$

Similarly

$$\begin{aligned} l\,(n, 3) &= l\,(n-1, 3) + l\,(n-1, 2) - l\,(n-2, 2) + l\,(n-3, 2) \\ &= l\,(n-1, 3) + \binom{n-3}{2} - \binom{n-4}{2} + \binom{n-5}{2} + (n-3) \\ &= l\,(n-1, 3) + (n-4) + \binom{n-5}{2} + (n-3) \\ &= l\,(n-1, 3) + \binom{n-4}{2}(n-2) \end{aligned}$$

which entails

$$l\,(n, 3) = \binom{n-2}{1} + 2\binom{n-3}{2} + \binom{n-4}{3}$$

By the mathematical induction, we get

$$l(n, k) = \sum_{i=0}^{\lambda} \binom{k-1}{i}\binom{n-k+1-i}{i+1} \text{ if } k \leqslant n \;\; \lambda = \min\left(k-1, \left(\frac{n-k}{2}\right)\right).$$

It is clear that $l(n, k) = 0$ if $k > n$, this completes the proof of the theorem.

Finally a table is given for the numbers $l(n, k)$ where $n = 0\,(1)\,12$.

The numbers $l(n, k)$

k/n	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9	10	11	12
2			1	2	4	7	11	16	22	29	37	46	56
3				1	2	5	10	18	30	47	70	100	138
4					1	2	6	13	26	48	83	136	213
5						1	2	7	16	35	70	131	232
6							1	2	8	19	45	96	192
7								1	2	9	22	56	126
8									1	2	10	25	68
9										1	2	11	28
10											1	2	12
11												1	2
12													1

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PSEUDO STRICT CONVEXITY AND METRIC CONVEXITY IN METRIC LINEAR SPACES

K. P. R. SASTRY[1], S. V. R. NAIDU[2] AND M. V. K. RAVI KISHORE[1]

[1]*Department of Mathematics, Andra University, Visakhapanam* 530003

[2]*Department of Applied Mathematics, A.U.P.G. Extension Centre, Nuzvid* 521201

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It is shown that a real linear metric with pseudo strict convexity and metric convexity induces a strictly convex norm. The case of a complex linear metric is also completly analysed.

Ahuja *et al.*[1] and Sastry and Naidu[3,4] studied extensively the mutual implications among various convexity conditions in metric linear spaces. In that connection, a natural question that arises is whether a metric linear space with pseudo strict convexity and metric convexity turns out to be a normed linear space. We answer this question in the affirmative in the real case and analyse the complex case. The remark on page 554 of Assad and Kirk[2] plays a key role in the proofs

Let (X, d) be a metric space. We say that d is metrically convex (m . c) if given $x \neq y$, there exists z ($\neq x$ and y) such that $d(x, z) + d(z, y) = d(x, y)$.

Suppose (X, d) is a metric space with metric convexity. Fix x, y in X. Write

$$S = \{z \in X / d(x, z) + d(z, y) = d(x, y)\}.$$

Clearly S is a closed set. For u, v in S, define $u \leqslant v$ if $d(x, v) = d(x, u) + d(u, v)$. Then $\leqslant$ is a partial order on S. Let A be a maximal totally ordered subset of S.

Then A has the following properties :

(I) $x, y \in A$ and A is closed.

(II) If $u, v \in A$ and $u < v$, then there exists $w \in A$ such that $u < w < v$.

(III) Suppose (X, d) is complete. Define $\varphi : A \to \mathrm{R}$ as $\varphi(z) = d(x, z)$.

Then φ is a strictly increasing continuous map of A onto $[0, d(x, y)]$. Further φ^{-1} is strictly increasing and continuous. Thus A is a perfect set. Infact, if $z \in A$ and $x < z \leqslant y$ then there exists a strictly increasing sequence in A converging to z; if $x \leqslant z < y$, then there exists a strictly decreasing sequence in A converging to z.

As a consequence of this, we have

(IV) (Assad and Kirk[2]). Suppose (X, d) is complete with metric convexity. If K is a closed subset of X, $x \in K$ and $y \notin K$, then there exists $z \in \partial K$ (boundary of K) such that $d(x, z) + d(z, y) = d(x, y)$.

The following example shows that the completion of a metrically convex metric space need not be metrically convex. It also shows that (III) and (IV) above may fail if the metric is not complete.

Example—Let $X = [0, 1/6] \cup (1/3, \frac{1}{2}] \cup (2/3, 1]$ with the usual metric on the real line.

Definition—Let (X, d) be a metric linear space. We say that

(i) d is pseudo strict convex (p.s.c) if $x \neq 0$, $y \neq 0$, $d(x + y, 0) = d(x, 0) + d(y, 0)$ imply that $y = tx$ for some $t > 0$.

(ii) d is ball convex (b.c) if $d(x, 0) = r = d(y, 0)$ implies that $d(\frac{1}{2}(x + y), 0) \leqslant r$ (that is, balls are convex).

(iii) d is strictly convex (s.c) if $r > 0$, $d(x, 0) \leqslant r$, $d(y, 0) \leqslant r$ imply that $d(\frac{1}{2}(x + y), 0) < r$ (that is, the balls are convex and do not contain line segments on the surface).

Lemma 1—Suppose d is a linear metric on R, the real line. Then d defines a norm if it is metrically convex.

PROOF : Suppose (R, d) is metrically convex. Suppose $0 < t < s$. Let $K = [-s, t]$. Then K is closed in (R, d), $0 \in K$ and $s \notin K$. Hence, by (IV), there exists $z \in \partial K$ such that $d(0, z) + d(z, s) = d(0, s)$. But $-s$ and t are the only points of ∂K. Hence $z = t$, so that $d(0, t) + d(t, s) = d(0, s)$. Thus, $d(0, x) + d(0, y) = d(0, x + y)$ for all non-negative x, y. From this we conclude that $d(0, t) = t\, d(0, 1)$ for all $t > 0$. Since $d(0, t) = d(0, -t)$, this leads to the conclusion.

Lemma 2—Suppose (X, d) is a real metric linear space. Then d defines a norm on X if and only if every one dimensional subspace of X has metric convexity.

PROOF : Applying Lemma 1 to one dimensional subspaces, the result follows.

Lemma 3—Let (X, d) be a real or complex metric linear space. Suppose that (X, d) has metric convexity and pseudo strict convexity. Then every one dimensional subspace of (X, d) has metric convexity.

PROOF : Let Y be a one dimensional subspace of (X, d). Suppose $x, y \in Y$ and $x \neq y$. Then, by metric convexity of (X, d), there exists z ($\neq x$ and y) such that $d(x, z) + d(z, y) = d(x, y)$. Hence $d(x - z, 0) + d(z - y, 0) = d(x - y, 0)$. Since (X, d) has (p.s.c), it follows that $z - y = t(x - z)$ for some $t > 0$, consequently z is in Y. Thus Y has metric covexity.

Theorem 4—Let (X, d) be a real metric linear space. Then d defines a strictly convex norm on X if and only if (X, d) has metric convexity and pseudo strict convexity.

Proof : If (X, d) has (m.c) and (p.s.c.), then, by Lemma 2 and Lemma 3, d defines a norm. Since, in a normed linear space, (s.c) and (p.s.c) are equivalent, it follows that d defines a strictly convex norm on X. The converse is evident.

The case of a complex linear metric is different. The following example shows that in a one dimensional complex metric linear space with (m.c) and (p.s.c.), the metric need not define a norm. In what follows, C is the field of complex numbers, regarded as one dimensional complex linear space.

Example 2—Define the linear metric d on the one dimensional complex linear space C as

$$d((0, 0), (a, b)) = (|a|^3 + |b|^3)^{1/3} \text{ for all } a, b \text{ in } R.$$

Then (C, d) has (m, c), (p.s.c) and (s.c) but d does not define a norm on C.

The following lemma characterizes norm-defining linear metrics on one dimensional complex linear spaces.

Lemma 5—Suppose d is a linear metric on the one dimensional complex linear space C. Then d defines a norm on C if and only if d is metrically convex and $d(z, 0) = d(|z|, 0)$ for all z in C, where $|z|$ is the usul absolute value of z (that is, metric is rotation invariant).

Proof : Suppose d is metrically convex and rotation invariant. Let $0 < t < s$. Write $K = \{z \in C \mid |z| \leqslant t\}$. Then K is closed in (C, d), $0 \in K$ and $s \notin K$, so that by (IV), there exists $w \in \partial K$ such that $d(0, w) + d(w, s) = d(0, s)$. Since $w \in \partial K$, we have $|w| = t$. Since $d(0, w) = d(0, |w|) = d(0, t)$, we have

$$d(0, t) + d(w, s) = d(0, s). \qquad \ldots(1)$$

Thus $d(0, t) < d(0, s)$ whenever $0 < t < s$.

Since $s - t = |s| - |w| \leqslant |s - w|$, we have

$d(0, s - t) \leqslant d(0, |s - w|) = d(0, s - w)$ so that $d(t, s) \leqslant d(w, s)$.

This, together with (1), gives $d(0, t) + d(t, s) = d(0, s)$.

Consequently, $d(0, a) + d(0, b) = d(0, a + b)$ for all non-negative a, b.

Now, using the continuity of d, we conclude that

$d(0, a) = a\, d(0, 1)$ for all non-negative a.

Finally, for $a, z \in C$,

$$d(0, az) = d(0, |az|) = |a||z| d(0, 1) = |a|\, d(0, |z|) = |a|\, d(0, z)$$

showing that d defines a norm on C.

The converse is evident.

The following theorem characterizes norm-defining linear metrics on complex linear spaces.

Theorem 6—Let (X, d) be a complex metric linear space. Then d defines a norm on X if and only if (i) every one dimensional subspace of (X, d) has metric convexity and (ii) $d(ax, 0) = d(|a|x, 0)$ for all $a \in C$ and $x \in X$. (condition (ii) is superfluous in real metric linear spaces).

PROOF : Suppose (i) and (ii) hold. Then Lemma 5 shows that d defines a norm on every one dimensional subspace of X. That is, d defines a norm on X. The converse is evident.

The following theorem is a consequence of Lemma 3 and Theorem 6.

Theorem 7—Let (X, d) be a complex metric linear space. Then d defines a strictly convex norm on X if and only if (i) (X, d) has (m.c) and (p.s.c) and

(ii) $d(ax, 0) = d(|a|x, 0)$ for all $a \in C$ and $x \in X$.

Lemma 8—Let (X, d) be a complete metric linear space with metric convexity and ball convexity. Then $d(x, z) + d(z, y) = d(x, y)$ for all x, y in X, where $z = \frac{1}{2}(x + y)$. In particular, every one dimensional subspace of (X, d) has metric convexity.

PROOF : Let $x, y \in X$. Write $r = \frac{1}{2} d(x, y)$. By (III), there exists $w \in A$ such that $d(x, w) = r$, so that $d(w, y) = r$. Now,

$$d(x, x + y - w) = d(w, y) = r = d(x, w) = d(y, x + y - w).$$

Thus w and $x + y - w$ are in both the balls $B(x, r)$, and $B(y, r)$ where $B(x, r) = \{u / d(x, u) \leqslant r\}$ and $B(y, r)$ has similar meaning. Since (X, d) has ball convexity, it follows that

$$z = \tfrac{1}{2}(w + 2z - w) \in B(x, r) \cap B(y, r).$$

Hence, $d(x, y) \leqslant d(x, z) + d(z, y) \leqslant r + r = d(x, y)$, which gives the result.

As a consequence of the above lemma, we have the following characterization of norm-defining linear metrics.

Theorem 9—Let (X, d) be a complete metric linear space. Then d defines a norm on X if and only if

(i) (X, d) has metric convexity and ball convexity and

(ii) $d(ax, 0) = d(|a|x, 0)$ whenever a is a scalar and $x \in X$.

(condition (ii) is superfluous in the real case).

The proof follows from Lemma 2, Theorem 6 and Lemma 8.

The following example shows that a metric linear space need not have ball convexity, even if every one dimensional subspace is strictly convex.

Example 3—For $(x, y) \in \mathbb{R}^2$, define

$$d((x, y), (0, 0)) = \begin{cases} \max\{|x|, |y|\} & \text{if } |x| + |y| \leqslant 1 \\ \max\{|x|, |y|\}. ((1 + |x| + |y|)/(2|x| + 2|y|)) & \end{cases}$$

otherwise. Then $(\mathbb{R}^2, d)$ is a real metric linear space which does not have any of (m c), (b.c) and (p.s.c). But every one dimesional subspace is unbounded and strictly convex.

The following is an example of a metric linear space without ball convexity, in which every one dimensional subspace has ball convexity.

Example 4—For $(x, y) \in \mathbb{R}^2$, define

$$d((x, y), (0, 0)) = \begin{cases} |x| + |y| & \text{if } |x| + |y| \leqslant \frac{1}{2} \\ \frac{1}{2} & \text{if } |x| + |y| \geqslant \frac{1}{2} \text{ and } |y| < \frac{1}{2} \text{ or if } 0 \leqslant |y| - \frac{1}{2} \leqslant |x| \\ \max\{|y| - |x|, 1\} & \text{if } |y| - |x| > \frac{1}{2}. \end{cases}$$

Then $(\mathbb{R}^2, d)$ has neither ball convexity nor (p.s.c), but every one dimensional subspace has ball convexity. While some of the one dimensional subspaces are bounded, some others are unbounded.

Problems : (1) Does a metric linear space with (m.c) have (b.c) ?

(2) Is the completion of a metric linear space with (m.c) and (b.c), metrically convex ?

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AN ANALOGUE OF HOFFMAN-WERMER THEOREM FOR A REAL FUNCTION ALGEBRA

S. H. KULKARNI AND N. SRINIVASAN

Department of Mathematics, Indian Institute of Technology, Madras 600036

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Let X be a compact Hausdorff space. Let $\tau : X \to X$ be a homeomorhpism such that τ^2 is the identity on X. Let $C(X)$ be the Banach algebra of all complex-Valued continuous functions on X. Let $C(X, \tau) = \{ f \in C(X) : f(\tau(x)) = \bar{f}(x)$ for all $x \in X\}$. In this paper the following characterization of $C(X, \tau)$ is proved :

If A is a real function algebra on (X, τ) and if Re A is uniformly closed, then $A = C(X, \tau)$.

1. INTRODUCTION

Let X be a compact Hausdorff space, $C(X)$ the Banach algebra of continuous complex-valued functions on X with the supremum norm and A, a function algebra on X (that is a uniformly closed subalgebra of $C(X)$ containing constants and separating points of X). The classical Stone-Weierstrass theorem gives a condition (namely that $f \in A$ implies $\bar{f} \in A$) which is necessary and sufficient for A to be the whole of $C(X)$.

Since the appearance of the Stone-Weierstrass theorem, several other conditions have been discussed. A systematic account of these can be found in Burckel[1]. Many of these conditions are in terms of Re A, the space of real parts of functions in A. One of these characterizations is given by the Hoffman-Wermer theorem which states that if Re A is uniformly closed, then $A = C(X)$. (Hoffman and Wermer[3]).

In the present note, we prove an analogue of the Hoffman-Wermer theorem for a real function algebra. This states that if X is a compact Hausdorff space, τ is an involutoric homeomorphism on X, A is a real function algebra on (X,τ) (see Definition 2.1), and Re A is uniformly closed, then

$$A = C(X, \tau) = \{ f \in C(X) : f(\tau(x)) = \bar{f}(x) \text{ for all } x \in X\}.$$

(Theorem 3.1). In proving this theorem, we have made use of an analogue of Bishop's theorem proved earlier by us[6] which deals with the partition of X into antisymmetric subsets. The next section contains the basic definitions and some properties of peak sets and antisymmetric sets. These properties are used in the last section to prove the

main theorem. This main theorem is then used to prove the following : Let A be a uniformly closed real subalgebra of $C(X)$, containing real constants and separating points of X. If Re A is closed, then there exists a closed subset Z of X such that

$$A = \{f \in C(X) : f \mid Z \text{ is real}\} \text{ (Theorem 3.7).}$$

This can be regarded as a stronger version of the classical Hoffman-Wermer Theorem (Remark 3.8).

We have undertaken a systematic study of the conditions on a real function algebra A on (X, τ) which force A to be the whole of $C(X, \tau)$. The present note contains one result in this direction. For other results along similar lines, see Kulkarni and Srinivasan[6,7].

2. Anti Symmetric Sets and Peak Sets

Let X be a compact-Hausdorff space and $C(X)$, the Banach algebra of all complex valued continuous functions on X with the supremum norm.

Let $\tau : X \to X$ be a homeomorphism such that $\tau^2 = \tau \circ \tau$ is the identity on X. Let

$$C(X, \tau) = \{f \in C(X) : f(\tau(x)) = \bar{f}(x) \text{ for all } x \in X\}.$$

Then $C(X, \tau)$ is a real communtative Banach algebra with the identity 1. Also it is easy to see that $C(X, \tau)$ separates the points of X, that is for any $x_1 \neq x_2$ in X there is f in $C(X, \tau)$ such that $f(x_1) \neq f(x_2)$.

Definition 2.1[5]—Let A be a subalgebra of $C(X, \tau)$. Then A is said to be a real function algebra on (X, τ) if (i) A is uniformly closed; (ii) A separates points of X; (iii) A contains (real) constants.

Note that, if $f \in A$ and f is constant, then f must be real.

Definition 2.2[6]—Let A be a real function algebra on (X, τ). A non empty subset K of X is called a set of anti-symmetry on A or A-antisymmetric if (i) $f \in A$ and $f \mid K$ is real implies $f \mid K$ is constant; and (ii) $f \in A$ and $f \mid K$ is purely imaginary implies $f \mid K$ is constant.

For the properties of A-antisymmetric sets we refer to the earlier paper of the authors cited above.

Definition 2.3—Let A be a real function algebra on (X, τ). Then A is said to be an anti-symmetric algebra if X is an A-antisymmetric set.

Definition 2.4—Let A be a real function algebra on (X, τ). Let S be a non-empty subset of X then S is said to be A-peak set if there exists $f \in A$ such that

$$S = \{x \in X : f(x) = 1\} \text{ and } |f(x)| < 1 \text{ for } x \in X - S.$$

We say that such a function f peaks on the set S.

The followings properties of A peak sets are elementary.

Property 2.5—If S is an A peak set then S is compact and $\tau(S) = S$.

Property 2.6—If $S = \bigcap_{n=1}^{\infty} S_n \neq \phi$ where each S_n is an A-peak set then S is also an A-peak set.

There exists $f_n \in A$ such that

$$S_n = \{x \in X : f_n(x) = 1\} \text{ and } |f_n(x)| < 1 \text{ on } X - S_n \text{ for all } n.$$

Let $f = \sum_{n=1}^{\infty} \frac{f_n}{2^n} \in A$. Clearly f peaks on S.

Definition 2.7—For $f \in A$ we define $\|f\|_y = \sup_{x \in y} |f(x)|$ for any subset y of x.

Property 2.8—Let $S = \bigcap_{\alpha \in \Lambda} S_\alpha \neq \phi$ where each S_α is an A-peak set and Y be a closed subset of X, such that $S \cap Y = \phi$. Then there exists $f \in A$, such that $\|f\| = 1$ $f|_S = 1$ and $\|f\|_y < 1$.

PROOF: $Y \cap S = \phi$. Hence $Y \subset S^c = \bigcup_{\alpha \in \Lambda} S_\alpha^c$ where S_α^c is open for all α. Therefore for some n

$Y \subset \bigcup_{i=1}^{n} S_{\alpha_i}^c$ implies $Y \cap (\bigcap_{i=1}^{n} S_{\alpha_i}) = \phi$. Now by 2.6 $\bigcap_{i=1}^{n} S_{\alpha_i}$ is an A peak set. Therefore exists $f \in A$ such that

$$\bigcap_{i=1}^{n} S_{\alpha_i} = \{x \in X : f(x) = 1\} \text{ and } |f(x)| < 1 \text{ on } X - \bigcap_{i=1}^{n} S_{\alpha_i}.$$

But

$$S \subset \bigcap_{i=1}^{n} S_{\alpha_i}$$

therefore $f(x) = 1$ on S.

$$\|f\| = 1 \quad |f(x)| < 1 \text{ on } X - \bigcap_{i=1}^{n} S_{\alpha_i} = \bigcap_{i=1}^{n} S_{\alpha_i}^c.$$

Since

$$Y \subset \bigcup_{i=1}^{n} S_{\alpha_i}^c$$

$|f(x)| < 1$ on Y. Hence $\|f\|_y < 1$

by compactness of Y.

The following theorem gives the relation between maximal A anti-symmetric sets and A peak sets. This theorem will be used to prove the main results of the paper.

Theorem 2.9—Let K be a maximal A antisymmetric set. Then $K \cup \tau(K)$ is the intersection of A peak sets that contain $K \cup \tau(K)$.

PROOF : Let $\mathscr{H}$ be the family of all A peak sets that contain $K \cup \tau(K)$. Since X is an A peak set and $K \cup \tau(K) \subset X$ we have $X \in \mathscr{H}$.

Thus $\mathscr{H}$ is non empty.

Let $Z = \bigcap_{S \in \mathscr{H}} S$. Note that $Z = \tau(z)$,

$K \cup \tau(K) \subset Z$. We want to show that $K \cup \tau(K) = Z$.

Claim 1—$f \in A$ and $f \mid z$ is real implies $f \mid z$ is constant. (The proof of this claim follows closely the proof of a similar theorem for a complex algebra given in Burckel[2]].

If not, there exists $f \in A$ such that $f \mid z$ is real and non constant.

Since K is an A-antisymmetric set $f \mid K$ is constant and hence $f \mid \tau(K) = \overline{f \mid K}$ is also constant. Thus $f \mid K \cup \tau(K)$ is a constant say a. Since $f \mid z$ is non-constant, there exists $Z_0 \in Z - (K \cup \tau(K))$ such that $f(z_0) = b \neq a$. Define $g_0 = 1 - \dfrac{(f-a)^2}{\|f-a\|_Z^2}$.

Then $g_0 \in A$.

$$g_0(Z) \subset [0, 1] \; g_0 \mid K \cup \tau(K) = 1 \text{ and } g_0(z_0) < 1.$$

Let

$$V_n = \{x \in X : |g_0(x)| < 1 + 1/2^n\}.$$

Then V_n is an open neighbourhood of $Z = \bigcap_{S \in \mathscr{H}} S$.

Hence $V_n^c \subset Z^c = \bigcup_{S \in \mathscr{H}} S^c$.

Thus $\{S^c : S \in \mathscr{H}\}$ is an open cover of the compact set V_n^c.

Hence $V_n^c \subset S_1^c \cup S_2^c \cup \ldots \cup S_m^c$ for some m i.e., $S_1 \cap S_2 \cap \ldots \cap S_m \subset V_n$

But $S_1 \cap S_2 \ldots \cap S_m = S$ is an A peak set by 2.6 and $K \cup \tau(K) \subset S$.

Hence $Z \subset S$.

Since S is an A-peak set there exists $g_n \in A$ such that

$$\|g_n\| = 1 \; g_n(S) = \{1\}$$

and

$$\|g_n\|_{X-V_n} < 1.$$

By taking sufficiently high power of g_n if necessary we may assume

$$\|g_n\|_{X-V_n} < \frac{1}{2^n \|g_n\|}.$$

Thus we have $\|g_n\| = 1$ $g_n(Z) = \{1\}$

and

$$\|g_n\|_{X-V_n} < \frac{1}{2^n \|g_0\|}. \qquad \text{...(1)}$$

Let

$$g = g_0 \sum_{n=1}^{\infty} \frac{g_n}{2^n}. \qquad \text{.. (2)}$$

Then

$$| g(x) | \leqslant | g_0(x) | \text{ for all } x \in X. \qquad \text{...(3)}$$

If $x \in \cap V_n$ then $| g_0(x) | < 1 + 1/2^n$ for all n that is $| g_0(x) | \leqslant 1$.

Thus by (3)

$$| g(x) | \leqslant 1 \text{ for all } x \in \cap V_n. \qquad \text{.. (4)}$$

Now suppose $x \in (X - \cap V_n)$.

Since V_n is a decreasing sequence ($V_{n+1} \subset V_n$) letting $V_0 = X$, there exists $m \geqslant 1$ such that $x \in V_{m-1}$ and $x \notin V_m$ then $x \notin V_n$ for all $n \geqslant m$.

Thus for $n = 1, 2, .. m - 1$

$$| g_0(x) g_n(x) | \leqslant 1 + \frac{1}{2^{m-1}}. \qquad \text{...(5)}$$

For $n > m$ since $x \notin V_n$

$$| g_0(x) g_n(x) | < 1/2^n < 1/2^{m-1}. \qquad \text{...(6)}$$

Combining (5) and (6)

$$| g(x) | \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} | g_n(x) g_0(x) |$$

$$\leqslant \left(1 + \frac{1}{2^{m-1}}\right) \sum_{n=1}^{m-1} \frac{1}{2^n} + \frac{1}{2^{m-1}} \sum_{n=m}^{\infty} \frac{1}{2^n} = 1$$

that is

$$| g(x) | \leqslant 1 \text{ for all } x \in X - \cap V_n. \qquad \text{...(7)}$$

(4), (7) implies $\|g\|_\infty \leqslant 1$.

Hence the function $\frac{(g+1)}{2} \in A$ peaks on the set $S = g^{-1}\{1\}$.

But $g_0 \mid K \cup \tau(K) = 1$ and $g_n \mid K \cup \tau(K) = 1$ for all n. Since $K \cup \tau(K) \subset Z$ and $g_n \mid z = 1$.

Hence $g(x) = g_0(x) \sum_{n=1}^{\infty} (1/2^n)\, g_n(x) = 1$ for all $x \in K \cup \tau(K)$.

Hence $K \cup \tau(K) \subset S$.

This means that S is an A peak set containing $K \cup \tau(K)$. Hence $S \in \mathscr{K}$. This implies that $Z \subset S = g^{-1}\{1\}$.

But $z_0 \in Z$ and $|g(z_0)| \leqslant |g_0(z_0)| < 1$ by (3) which is a contradiction.

This proves the claim.

Now if $f \in A, f \mid z$ is purely imaginary implies $f \mid z$ is constant and Z is an A antisymmetric set. But since K as well as $\tau(K)$ are maximal A-antisymmetric sets (Kulkarni and Srinivasan[6], Lemma 2.10) contained in Z we have $K = Z = \tau(K)$. This proves the theorem.

If not, there exists $h \in A$ such that $h \mid z$ is purely imaginary and non constant. But then $h^2 \mid z$ is real and must be constant by claim 1. We may assume $h^2 \mid z = -1$.

Thus $h(x) = \pm i$ for all $x \in Z$.

Let

$$Z_1 = \{x \in Z : h(x) = i\}$$

and

$$Z_2 = \{x \in Z : h(x) = -i\}.$$

Then $Z = Z_1 \cup Z_2$, $Z_1 \cap Z_2 = \phi$ and Z_1 and Z_2 are nonempty. $\tau(Z_1) = Z_2$ and $\tau(Z_2) = Z_1$. Since $h \mid K$ must be constant, $h \mid K = i$ or $h \mid K = -i$. Hence $K \subset Z_1$ or $K \subset Z_2$.

Claim 2—Z_1 is an A-antisymmetric set. Let $f \in A$ and $f \mid Z_1$ be real. But then $f \mid Z_2 = \overline{(f \mid Z_1)}$ is also real, and hence $f \mid z$ is real and by Claim 1, $f \mid z$ is constant. Hence $f \mid Z_1$ is constant.

Now let $f \in A$ and $f \mid Z_1$ be purely imaginary. But then $fh \mid Z_1 = if \mid Z_1$ is real and hence constant. This proves $f \mid Z_1$ is constant. Thus claim 2 is proved.

If $K \subset Z_1$ by maximality of K, $K = Z_1$, $\tau(K) = Z_2$. If $K \subset Z_2$, $K = Z_2$, $\tau(K) = Z_1$.

In either case, $K \cup \tau(K) = Z_1 \cup Z_2 = Z$. This completes the proof of the theorem.

Corollary 2.10—Let A be a real function algebra on $(X, \tau,)$ and K be a maximal A-antisymmetric set. Then $A \mid K$ is uniformly closed in $C(K)$ (see Theorem 3.6 of Kulkarni and Srinivasan[6] for a different proof).

PROOF: For $f \in A$ and $\epsilon > 0$ we can find $F \in A$ such that $\|F\|_X < \|f\|_K + \epsilon$ and $F = f$ on K.

We may assume that $\|f\|_X \leqslant 1$.

Let

$$V = (x \in X : | f(x) | < \|f\|_K + \epsilon\}.$$

Clearly V is a neighbourhood of $K \cup \tau(K)$. Hence by Theorem 2.9 and Property 2.8, there exists $g \in A$ such that $\|g\| = 1$ $g \mid K \cup \tau(K) = 1$ and $\|g\|_{X-V} < 1$.

By taking sufficiently high power of g, if necessary, we many assume

$$\|g\|_{X-V} < \|f\|_K + \epsilon.$$

Let $F = fg \in A$. Clearly $F = f$ on K.

For

$$x \in V, | F(x) | = | f(x) | g(x) | < \|f\|_K + \epsilon \text{ as}$$
$$| f(x) | < \|f\|_K + \epsilon.$$

For $x \in X - V$,

$$| F(x) | = | f(x) \| g(x) | < \|f\|_K + \epsilon \text{ as}$$
$$| g(x) | < \|f\|_K + \epsilon.$$

Thus

$$\|F\|_X < \|f\|_K + \epsilon.$$

This proves the claim.

Now let

$$A_K = \{f \in A; f \mid K = 0\}$$

and $Q = A \mid A_K$ the quotient space. Q is complete in the quotient norm.

For a typical element $g + A_K$ of Q, where $g \in A$

$$\begin{aligned} \|g + A_K\| &= \inf_{f \in A_K} \|g + f\| \\ &= \inf \{ \|h\| : h \mid K = g \mid K\} \\ &= \|g\|_K \text{ by the above claim.} \end{aligned}$$

Thus $A \mid K$ is isometric to Q. Hence $A \mid K$ is also complete and hence closed in $C(K)$.

3. A Characterization of $C(X, \tau)$

We present a proof of the main theorem in this section using the properties discussed in Section 2.

Theorem 3.1—Let A be a real function algebra on (X, τ). Let $\text{Re } A = \{\text{Re } f : f \in A\}$. If Re A is uniformly closed then $A = C(X, \tau)$.

We shall need the following Lemmas in the proof of the above theorem.

Lemma 3.2—Let A be a real function algebra on (X, τ). If Re A is closed and A is antisymmetric then X is a singleton set.

Proof : Since A is an anti-symmetric algebra, by Definition 2.3 we have the following :

(i) $f \in A$ and f is real implies f is constant and

(ii) $f \in A$ and f is purely imaginary implies that f is constant.

Let $x_0 \in X$. Since A is antisymmetric, for every $u \in \text{Re } A$ there exists unique $f \in A$ such that $u = \text{Re } f$.

Define $T(u) = f$. Then $T : \text{Re } A \to A$ is a linear map. Let $u_n \in \text{Re } A$, $u_n \to u$ and $T(u_n) = f_n \to f$. Then clearly $\|\text{Re } f_n - \text{Re } f\|_\infty \leqslant \|f_n - f\| \to 0$

that is $u_n \to \text{Re } f$ and hence $u = \text{Re } f$.

Thus $T(u) = f$. This shows that T has a closed graph. Hence T is a bounded operator. Suppose X contains more than one point.

Let $g = u + iv \in A$ be a non-constant function such that $g(x_0) = 0$.

Let

$$R = \left\{ x + iy \in C \;\middle|\; \begin{array}{l} -\|u\| \leqslant x \leqslant \|u\| \\ -\|v\| \leqslant y \leqslant \|v\| \end{array} \right\}.$$

Since g is non constant $V \neq 0$. Hence there exists $x_1 \in X$ such that $|V(x_l)| = \|V\|$.

Let $Z_1 = g(x_1)$. Then Z_1 and $\bar{Z}_1$ are boundary points of R.

Let

$$N > 0 \text{ and } R_N = \left\{ \begin{array}{l} x + iy \in C : -\|u\| \leqslant x \leqslant \|u\| \\ \quad \text{and } -N \leqslant y \leqslant N \end{array} \right\}.$$

[Note that R and R_N are symmetric w.r.t. real axis, that is $w \in R$ (respectively R_N) if and only if $\bar{w} \in R$ (respectively R_N]. There exists a conformal map $\phi : R^0 \to R_N^0$ such that

$$\phi(z_1) = iN \text{ and } \phi(\bar{z}_1) = -iN.$$

This map extends to a homeomorphism of boundaries of R and R_N.

Thus ϕ is a uniform limit of polynomials in R.

Define $\psi : R \to R_N$ as $\psi(z) = \phi(z) + \overline{\phi(\bar{z})}$.

Then ψ is a uniform limit of polynomials with real coefficients because if $\{p_n(z)\}$ is a sequence of polynomials converging uniformly to $\phi(z)$ on R then $q_n(z) = p_n(z) + \overline{p_n(\bar{z})}$ is a sequence of polynomials with real coefficients converging uniformly to $\psi(z)$.

Therefore $h = \psi(g) \in A$ by spectral mapping theorem[4]. Now $T(\mathrm{Re}\, h) = h$.

$$h(x_1) = \psi(g\, x_1)) = \psi(z_1) = \phi(z_1) + \overline{\phi(\bar{z}_1)}$$

$$= iN + \overline{(-iN)} = 2iN.$$

Thus we have $\|\mathrm{Re}\, h\| \leqslant \|u\|$

$$\|h\| \geqslant N.$$

Since N is arbitrary this is a contradiction to the fact that T is a bounded linear map. Therefore X is a singleton set.

Lemma 3.3—Let A be a real function algebra on (X, τ) and suppose $\mathrm{Re}\, A$ is uniformly closed. Let S be a maximal A antisymmetric set. Then

$\mathrm{Re}\, A_S = \{\mathrm{Re}\, f \mid s : f \in A\}$ is also uniformly closed in $C(S)$.

PROOF : *Claim* 1—If $f \in A$ and $\epsilon > 0$ we can find $F \in A$ such that $\|\mathrm{Re}\, F\|_X \leqslant \|\mathrm{Re}\, f\|_S + 2\epsilon$

and $Re\ F = \mathrm{Re}\, f$ on S.

We may assume that $\|f\| \leqslant 1$.

Let

$$\Omega = \{w = s + it \in C : |w| < 1 \text{ and } |t| < \epsilon\}.$$

Let

$$U = \{z \in C : |z| < 1\}.$$

(Note that both Ω and U are symmetric w.r.t. real axis) and ν be a conformal map of U on Ω with $\gamma(0) = 0$ and $\nu(1) = 1$.

Now ν is analytic in U.

Define $\mu : U \to \Omega$ as

$$\mu(z) = \frac{(\nu(z) + \overline{\nu(\bar{z})})}{2}.$$

Then μ is analytic in U. $\mu(0) = 0$ and $\mu(1) = 1$.

Choose $\delta > 0$ such that $|z| < \delta$ implies $|\mu(z)| < \epsilon$.

Since S is a maximal A antisymmetric set, by Theorem 2.9, $S \cup \tau(S) = \bigcap_{\alpha \in \wedge} A_\alpha$ where each A_α is an A-peak set.

Let

$$V = \{x \in X : |\operatorname{Re} f(x)| < \|\operatorname{Re} f\|_S + \epsilon\}.$$

Then V is a neighbourhood of S.

Since $\operatorname{Re} f(x) = \operatorname{Re} f(\tau(x))$ for every $x \in X$, we have $V = \tau(V)$. Thus $S \cup \tau(S) \subset V$. Therefore by 2.8 there exists $g \in A$ such that $\|g\| = 1$

$$g \mid S \cup \tau(S) = 1 \text{ and } \|g\|_{X-U} < 1.$$

Choose a positive integer n large enough such that

$$\|g^n\|_{X-U} < \delta.$$

Let $h = \mu(g^n)$. Then $h \in A$ by the spectral mapping theorem for real Banach algebras[4].

$h = 1$ on $S \cup \tau(S)$

$|\operatorname{Im} h| \leqslant \epsilon$ on all of X.

Also $|\operatorname{Re} h| < \epsilon$ on $X - U$ and $|\operatorname{Re} h| \leqslant 1$ on all of X. Let $F = fh \in A$.

$$\operatorname{Re} F = \operatorname{Re} f \operatorname{Re} h - \operatorname{Im} f \operatorname{Im} h.$$

$h = 1$ on S implies $\operatorname{Re} h = 1$, $\operatorname{Im} h = 0$ on S

$$\operatorname{Re} F = \operatorname{Re} f \text{ on } S.$$

On

$$U \; |\operatorname{Re} F| \leqslant |\operatorname{Re} f| \, |\operatorname{Re} h| + |\operatorname{Im} f| \, |\operatorname{Im} h|$$
$$\leqslant (\|\operatorname{Re} f\|_S + \epsilon). \;\; 1 + \epsilon.$$

On

$$X - U, \; |\operatorname{Re} F| \leqslant \epsilon + \epsilon.$$

Thus

$$\|\operatorname{Re} F\|_X \leqslant \|\operatorname{Re} f\|_S + 2\epsilon$$

and

$$\operatorname{Re} F = \operatorname{Re} f \text{ on } S.$$

To prove Re A_S is closed : Let $R_S = \{u \in \operatorname{Re} A : u(S) = 0\}$ Re A with the supremum norm is a Banach space and R_S is a closed subspace of Re A.

Consider the quotient space

$$Q = \operatorname{Re} A/R_S = \{\operatorname{Re} f + R_S : f \in A\}.$$

Then Q is complete in the quotient norm

$$\|\operatorname{Re} f + R_S\| = \inf \{\|\operatorname{Re} F\| : F \in A \text{ and } \operatorname{Re} F = \operatorname{Re} f \text{ on } S\}$$

$$= \|\operatorname{Re} f\|_S \text{ by claim 1.}$$

Thus Re A_S is isometrically isomorphic to Q and hence complete in the maximum norm on S.

This proves that Re A_S is closed.

We make use of the following theorem in proving Theorem 3.1.

Bishop's Theorem[6]—Let A be a real function algebra on (X, τ). Suppose $f \in C(X, \tau)$ is such that $f \mid_K \in A \mid_K$ for every maximal A-antisymmetric subset K of X, then $f \in A$.

Proof of Theorem 3.1—Let K be a maximal A-antisymmetric set. We shall show that K is singleton.

Denote the restriction algebra $A \mid_K$ by A_K. Note that A_K is uniformly closed in $C(K)$ by Corollary 2.10. By Corollary 2.11 of Kulkarni and Srinivasan[6], either $K = \tau(K)$ or $K \cap \tau(K) = \phi$.

Case (*i*) $K = \tau(K)$: In this case, A_K can be regarded as a real function algebra on $(K, \tau \mid_K)$. Obviously A_K is an antisymmetric algebra. Re A_K is uniformly closed by Lemma 3.3. Hence by Lemma 3.2, K is singletone.

Case (*ii*) $K \cap \tau(K) = \phi$: In this case it can easily be shown that there exists $h \in A$ such that $h \mid_K = i$ (see Theorem 2.17 of Kulkarni[6]). This means $i \in A_K$. Thus A_K can be regarded as a complex function algebra on K. As before, A_K is an antisymmetric algebra and Re A_K is uniformly closed. Hence K is singleton by the main lemma in Hoffman and Wermer[3].

Thus every maximal A-antisymmetric set is singleton. Hence by the above analogue of Bishop's theorem $A = C(X, \tau)$.

Corollary 3.4—If Re $A \supset C_R(X, \tau)$, then $A = C(X, \tau)$.

Corollary 3.5 (Stone-Weierstrass)—Let A be a real function algebra on (X, τ) such that $\bar{f} \in A$ for all $f \in A$ then $A = C(X, \tau)$.

Proof : Since A is closed under complex conjugation Re A is the collection of all real valued functions in A. Hence Re A is closed.

Remark 3.6 : Mehta[8] has defined a real function algebra on a compact Hausdorff space X as a uniformly closed real subalgebra of $C(X)$ that contains the constant

function 1 and separates the points of X. If A is such an algebra and Re A is closed, then what can be said about A ? The following theorem answers this question.

Theorem 3.7—Let X be a compact Hausdorff space and A a uniformly closed real subalgebra of $C(X)$, which contains (real) constants and which separates points of X. If Re A is uniformly closed in $C(X)$, then there exists a closed subset Z of X such that

$$A = \{f \in C(X) : f \mid_Z \text{ is real}\}.$$

PROOF : Let $Z = \{x \in X : f(x) \text{ is real for all } f \in A\}$. Let $Y = X \times \{0, 1\}$. For every $z \in Z$ we identify the points $(z, 0)$ and $(z, 1)$ and denote the space obtained by this identification by $\tilde{X}$. (We may consider $\tilde{X}$ as the space obtained by pasting together two copies of X at the points of Z). Define $\tau : Y \to Y$ as $\tau((x, 0)) = (x, 1)$ and $\tau((x, 1)) = (x, 0)$ for every $x \in X$. Let $\tilde{\tau}$ be the map induced by τ on $\tilde{X}$. It is easy to see that $\tilde{X}$ is a compact Hausdorff space and $\tilde{\tau}$ is an involutoric homeomorphism on $\tilde{X}$. We may regard X as a subset of $\tilde{X}$. Then every point of Z is a fixed point of $\tilde{\tau}$.

Now Let $f \in C(X)$ and f be real on Z.

Define

$$\tilde{f} \text{ on } \tilde{X} \text{ as } \tilde{f}(x, 0)) = f(x)$$

$$\tilde{f}((x, 1)) = \bar{f}(x)$$

for all $x \in X$.

Since for $x \in Z$, $(x, 0)$ and $(x, 1)$ are identified and $f(x)$ real, $\tilde{f}$ is well-defined.

Clearly $\tilde{f} \in C(\tilde{X}, \tilde{\tau})$ and $\tilde{f} \mid X = f$.

Let $\tilde{A} = \{\tilde{f} : f \in A\}$. Then it is routine to check that $\tilde{A}$ is a real function algebra on $(\tilde{X}, \tilde{\tau})$ and Re $\tilde{A}$ is uniformly closed. Hence by Theorem 3.1, $\tilde{A} = C(\tilde{X}, \tilde{\tau})$. This in turn, implies that $A = \{f \in C(X) : f \text{ is real on } Z\}$.

Remark 3.8 : As special cases of the above theorem, we get :

If $Z = \phi$, then $A = C(X)$ and if $Z = X$, then $A = C_R(X)$. In particular, if for every $x \in X$ there exists $f \in A$ such that $\operatorname{Im} f(x) \neq 0$, then $Z = \phi$ and hence $A = C(X)$. This is a stronger version of the classical Hoffman-Wermer theorem.

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ON THE APPROXIMATION OF ANALYTIC FUNCTIONS REPRESENTED BY DIRICHLET SERIES

G. S. SRIVASTAVA AND SUNITA RANI

Department of Mathematics, University of Roorkee, Roorkee

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Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ represent an analytic function in the half plane Re $s < \alpha$, $(-\infty < \alpha < \infty)$. Let $E_n(f, \beta)$ be the error in approximating the function $f(s)$ by exponential polynomials in the half plane Re $s \leqslant \beta < \alpha$. In this paper, the lower order and lower type of $f(s)$ have been characterized in terms of the rate of decrease of $E_n(f, \beta)$. Further, the order, type, lower order and lower type of $f(s)$ have been obtained in terms of the ratio of consecutive error terms $E_{n-1}(f, \beta)/E_n(f, \beta)$.

§ 1. Let

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \quad \ldots(1.1)$$

where $0 < \lambda_n < \lambda_{n+1} \to \infty$, $s = \sigma + it$ (σ and t being real variables), $[a_n]_1^{\infty}$ is a sequence of complex numbers, be a Dirichlet series. We assume that

$$\lim_{n\to\infty} \inf(\lambda_{n+1} - \lambda_n) = 0 > 0. \quad \ldots(1.2)$$

Then we have

$$\lim_{n\to\infty} \sup (n/\lambda_n) = D < \infty. \quad \ldots(1.3)$$

If the series (1.1) converges absolutely in the half plane $\sigma =$ Re $s < \alpha$ then [2] $f(s)$ is an analytic function for $\sigma < \alpha$ and

$$\alpha = \lim_{n\to\infty} \sup \frac{\log |a_n|}{\lambda_n}. \quad \ldots(1.4)$$

For any $\sigma < \alpha$, we set $M(\sigma) = \underset{-\infty<t<\infty}{\text{l. u. b.}} |f(\sigma + it)|$. The growth of the analytic function $f(s)$ is studied through the parameters such as order, type etc. Thus $f(s)$ is said io be [1] of order ρ if

$$\lim_{\sigma\to\alpha} \sup \frac{\log\log M(\sigma)}{-\log[1 - \exp(\sigma - \alpha)]} = \rho, 0 \leqslant \rho \leqslant \infty. \quad \ldots(1.5)$$

Further, if $0 < \rho < \infty$ then the type T of $f(s)$ is defined as

$$\limsup_{\sigma \to \alpha} \frac{\log M(\sigma)}{[1 - \exp(\sigma - \alpha)]^{-\rho}} = T, \; 0 \leqslant T \leqslant \infty. \quad \ldots(1.6)$$

Likewise, the lower order λ and lower type t $(0 < \rho < \infty)$ of $f(s)$ are defined as[4,5],

$$\liminf \frac{\log \log M(\sigma)}{-\log [1 - \exp(\sigma - \alpha])} = \lambda, \; 0 \leqslant \lambda \leqslant \rho \leqslant \infty \quad \ldots(1.7)$$

$$\liminf_{\sigma \to \alpha} \frac{\log M(\sigma)}{[1 - \exp(\sigma - \alpha)]^{-\rho}} = t, \; 0 \leqslant t \leqslant \infty. \quad \ldots(1.8)$$

Let $\mu(\sigma) = \max\limits_{n \geqslant 1} \{ | a_n | \exp(\sigma \lambda_n)\}$ be the maximum term in the expansion (1.1) and $\lambda_{N(\sigma)} = \max\limits_{n} \{\mu(\sigma) = | a_n | \exp(\sigma \lambda_n)\}$ denote the rank of the maximum term. For an analytic function $f(s)$ of finite order, it is known Nandan[3] that $\log M(\sigma) \simeq \log \mu(\sigma)$ as $\sigma \to \alpha$. Hence $M(\sigma)$ can be replaced by $\mu(\sigma)$ in the definitions (1.5) to (1.8). Nandan[3] proved that if $f(s)$ is of lower order λ then

$$1 + \lambda = \liminf_{\sigma \to \alpha} \frac{\log \lambda_{N(\sigma)}}{-\log [1 - \exp(\sigma - \alpha)]} \; . \quad \ldots(1.9)$$

However, it was shown[8] by means of a counter example that the result (1.9) does not always hold. We shall call an analytic function $f(s)$ to be of "admissible class" if (1.9) holds.

Let D_α denote the class of all functions given by (1.1) which are analytic for $\sigma < \alpha$ and let $\bar{D}_\beta$, $-\infty < \beta < \alpha < \infty$, denote the class of all functions given by (1.1) which are analytic for $\sigma \leqslant \beta$. The notion of approximation of an analytic function by means of exponential polynomials was introduced by Nautiyal and Shukla[7]. In the series (1.1), if $a_n = 0$ for $n > k$ and $a_k \neq 0$ then $f(s)$ is said to be an exponential polynomial of degree k. The class of all exponential polynomials of degree almost k is denoted by π_k. Following Nautiyal and Shukla[7] we define

$$E_n(f, \beta) = \inf_{p \in \pi_n} \|f - p\|_\beta, \; n = 0, 1, 2, \ldots \quad \ldots(1.10)$$

where $\|f - p\|_\beta = \text{l. u. b.}_{-\infty < t < \infty} | f(\beta + it) - p(\beta + it) |$. Nautiyal and Shukla obtained the characterizations of ρ and T in terms of the rate of decay of the approximation error $E_n(f, \beta)$. Hence (Nautiyal and Shukla[7], Theorem 2 and 3) we have

Theorem A—Let $f(s) \in D_\alpha$ be of order ρ and $-\infty < \beta < \alpha < \infty$. Then

$$\rho = \limsup_{n \to \infty} \frac{\log^+ \log^+ [E_n(f, \beta) \exp\{(\alpha - \beta) \lambda_{n+1}\}]}{\log \lambda_{n+1} - \log^+ \log^+ [E_n(f, \beta) \exp\{(\alpha - \beta) \lambda_{n+1}\}]} \; . \quad \ldots(1.11)$$

Theorem B—Let $f(s) \in D_\alpha$ and $-\infty < \beta < \alpha < \infty$. Then $f(s)$ is of order ρ $(0 < \rho < \infty)$ and type T, if and only if

$$\left(\frac{\rho+1}{\rho}\right)^{\rho+1} (\rho T) = \limsup_{n\to\infty} \frac{[\log^+\{E_n(f,\beta)\exp((\alpha-\beta)\lambda_{n+1})\}]^{\rho+1}}{(\lambda_{n+}{}^1)^\rho} \quad \ldots(1.12)$$

when the right hand side of (1.12) is a non-zero finite number.

In the present paper, we obtain the characterizations for the lower order λ and lower type t in terms of the decay of the approximatione rror $E_n(f,\beta)$. We also obtain these characterization in terms of the ratio of consecutive errors $E_{n-1}(f,\beta)/E_n(f,\beta)$. To avoid repititions, we shall assume throughout that $f(s) \in D_\alpha$ and the condition (1.2) is satisfied. Further, we shall write E_n for the error term $E_n(f,\beta)$.

§2. We now prove

Lemma—Let $f_\beta(s) = \sum_{n=1}^{\infty} \{E_{n-1}\exp(-\beta\lambda_n)\}\exp(s\lambda_n)$. Then $f_\beta(s) \in D_\alpha$.

Further,

$$\left.\begin{array}{l}\lambda \leqslant \lambda(f_\beta),\ \rho = \rho(f_\beta);\\ T = T(f_\beta),\ t = t(f_\beta),\ 0 < \rho < \infty\end{array}\right\}. \quad \ldots(2.1)$$

PROOF : That $f_\beta(s) \in D_\alpha$ follows from (Nautiyal and Shukla[7], Theorem 1] and (1.4). Consequently, we have $\rho = \rho(f_\beta)$ from (1.11) and Theorem 1 of Juneja and Nandan[1]. From Lemma 1 of Nautiyal and Shukla[7], we have

$$E_{n-1} \leqslant K\,M(\sigma)/\exp((\sigma-\beta)\lambda_n),\ n = 1, 2, \ldots \quad \ldots(2.2)$$

where K is a constant independent of n and σ. Hence we have

$$\mu(\sigma, f_\beta) \leqslant K\,M(\sigma).$$

Also from (3.6) of Nautiyal and Shukla[7], we have

$$M(\sigma) \leqslant |\,a_0\,| + 2M(\sigma, f_\beta). \quad \ldots(2.4)$$

Hence from (2.4) we have $\lambda \leqslant (f_\beta)$. Further for $0 < \rho < \infty$, (2.3), (2.4) and the definitions of $T(f_\beta)$, T etc. lead to $T = T(f_\beta)$, $t = t(f_\beta)$. This proves the lemma.

In the following results, we denote

$$\log^+ x = \max(0, \log x).$$

Theorem 1—Let $f(s)$ be of order $\rho > 0$ and lower order λ, $0 \leqslant \lambda \leqslant \infty$.

Then

$$1+\lambda \geqslant \liminf_{n\to\infty} \frac{\log\lambda_n}{\log\lambda_{n+1} - \log^+\log^+[E_n\exp\{(\alpha-\beta)\lambda_{n+1}\}]}. \quad \ldots(2.5)$$

Further, if $\psi(n) = \log(E_{n-1}/E_n)/(\lambda_{n+1} - \lambda_n)$ forms a non decreasing function of n for all large values of n and $f_\beta(s)$ belongs to 'admissible class', then

$$1+\lambda \leqslant \liminf_{n\to\infty} \frac{\log \lambda_n}{\log \lambda_{n+1} - \log^+ \log^+ [E_n \exp\{(\alpha-\beta)\lambda_{n+1}\}]} . \qquad \ldots(2.6)$$

PROOF : Let the right hand side expression in (2.5) be denoted by θ. We assume that $1 < \theta < \infty$. For a given $\epsilon > 0$ such that $0 < \epsilon < \theta$, there exists integer $N = N(\epsilon)$ such that

$$\log E_n + (\alpha - \beta)\lambda_{n+1} > \lambda_{n+1}(\lambda_n)^{-1/(\theta-\epsilon)}\ n > N(\epsilon).$$

From (2.2), we have for all σ close to α,

$$\log M(\sigma) \geqslant \log E_k + (\sigma - \beta)\lambda_{k+1} + O(1),\ k = 1, 2, \ldots \qquad \ldots(2.7)$$

Hence we have

$$\log M(\sigma) > \lambda_{n+1}\ (\lambda_n)^{-1/(\theta-\epsilon)} + (\sigma - \alpha)\lambda_{n+1},\ n > N(\epsilon).$$

We choose a sequence $[\sigma_n]$ by defining

$$\alpha - \sigma_n = (1/e)(\lambda_n)^{-1/(\theta-\epsilon)},\ n \geqslant N.$$

It is evident that $\sigma_n < \alpha$ and $\sigma_n \to \alpha$ as $n \to \infty$. For $\sigma_n \leqslant \sigma < \sigma_{n+1}$, we thus have

$$\begin{aligned} \log M(\sigma) &> \lambda_{n+1}(\lambda_n)^{-1/(\theta-\epsilon)} + (\sigma_n - \alpha)\lambda_{n+1} \\ &= (1 - e^{-1})\lambda_{n+1}(\lambda_n)^{-1/(\theta-\epsilon)} \\ &= e(1 - e^{-1})(\alpha - \sigma_n)[e(\alpha - \sigma_{n+1})]^{-(\theta-\epsilon)} \\ &> (e - 1)(\alpha - \sigma)^{1-(\theta-\epsilon)} e^{-(\theta-\epsilon)}. \end{aligned}$$

Therefore

$$\log\log M(\sigma) > O(1) + (1 - \theta + \epsilon)\log(\alpha - \sigma).$$

It can be easily seen that $1 - \exp(\sigma - \alpha) \simeq \alpha - \sigma$ as $\sigma \to \alpha$. Hence on proceding to limits, we obtain

$$\lambda = \liminf_{\sigma\to\alpha} \frac{\log\log M(\sigma)}{-\log[1 \exp -(\sigma - \alpha)]} \geqslant \theta - 1.$$

The result is trivial if $\theta \leqslant 1$. If $\theta = \infty$, then the result follows on taking an arbitrarily large number in place of $(\theta - \epsilon)$ and proceeding as above. This proves (2.5).

To prove (2.6), we consider $f_\beta(s) = \sum_{n=1}^{\infty} b_n \exp(s\lambda_n)$ where $b_n = E_{n-1}\exp(-\beta\lambda_n)$.

Then

$$\phi(n) = \frac{\log | b_n/b_{n+1} |}{\lambda_{n+1} - \lambda_n} = \frac{\log(E_{n-1}/E_n)}{\lambda_{n+1} - \lambda_n} + \beta = \psi(n) + \beta. \qquad \ldots(2.8)$$

Hence $\phi(n)$ is a non-decreasing function of n whenever $\psi(n)$ satisfies the condition. Applying Lemma 3 of Nandan[4] to the function $f_\beta(s)$ (under the assumption that (1.9) is satisfied for $f_\beta(s)$) we obtain on using (2.1),

$$1+\lambda \leqslant 1+\lambda(f_\beta) = \liminf_{n\to\infty} \frac{\log \lambda_{n-1}}{\log \lambda_n - \log^+ \log^+ [E_{n-1} \exp\{(\alpha-\beta)\lambda_n\}]}.$$

This proves (2.6) and the proof of Theorem 1 is complote. Next we prove

Theorem 2—Let $f(s)$ be of order ρ, $0<\rho<\infty$, and lower type t, $0 \leqslant t \leqslant \infty$, Then

$$\left(\frac{\rho+1}{\rho}\right)^{\rho+1} (\rho t) \geqslant \liminf_{n\to\infty} \lambda_n \left[\frac{\log^+ \{E_n \exp(\alpha-\beta)\lambda_{n+1})\}}{\lambda_{n+1}}\right]^{\rho+1}. \qquad \ldots(2.9)$$

Further, if $\psi(n)$ as defined before, forms a non-decreasing function of n for all large values of n, then

$$\left(\frac{\rho+1}{\rho}\right)^{\rho+1} (\rho t) \leqslant \liminf_{n\to\infty} \lambda_{n+1} \left[\frac{\log^+ \{E_n \exp((\alpha-\beta)\lambda_{n+1})\}}{\lambda_{n+1}}\right]^{\rho+1}. \qquad \ldots(2.10)$$

PROOF : Let the right hand side expression of (2.9) be denoted by A, $0<A<\infty$. Then for a given ϵ, arbitrarily small and satisfying $0<\epsilon<A$, we have

$$\lambda_n \left[\frac{\log E_n + (\alpha-\beta)\lambda_{n+1}}{\lambda_{n+1}}\right]^{\rho+1} > A-\epsilon,\ n > N = N(\epsilon)$$

or

$$\log E_n > \lambda_{n+1} [\{(A-\epsilon)/\lambda_n\}^{1/(\rho+1)} - (\alpha-\beta)],\ n > N.$$

On using (2.7) we have

$$\log M(\sigma) > \lambda_{n+1} [\{(A-\epsilon)/\lambda_n\}^{1/(\rho+1)} - (\alpha-\sigma)] + O(1),\ n > N.$$

We define a sequence $[\sigma_n]$ by putting

$$\alpha - \sigma_n = \left(\frac{\rho}{\rho+1}\right)\left(\frac{A-\epsilon}{\lambda_n}\right)^{1/(\rho+1)},\ n = N+1, \ldots.$$

Then for $\sigma_n \leqslant \sigma < \sigma_{n+1}$, we have

$$\log M(\sigma) > \lambda_{n+1} [(A-\epsilon)/\lambda_n]^{1/(\rho+1)} - (\alpha-\sigma_n)\lambda_{n+1},\ n > N$$

$$> \frac{1}{\rho}\left(\frac{\rho}{\rho+1}\right)^{\rho+1} (A-\epsilon)(\alpha-\sigma)^{-\rho}, \text{ as in Theorem 1.}$$

Hence we get

$$t = \liminf_{\sigma\to\alpha} \frac{\log M(\sigma)}{[1-\exp(\sigma-\alpha)]^{-\rho}} \geqslant \frac{A}{\rho}\left(\frac{\rho}{\rho+1}\right)^{\rho+1}$$

and (2.9) follows. The proof of (2.9) when $A = \infty$ can be completed as before.

On using the condition on the function $\psi(n)$, we obtain for analytic function $f_\beta(s)$. [Nandan[5], Nandan and Srivastava[6] (Th.[4])]

$$\left(\frac{\rho+1}{\rho}\right)^{\rho+1} \rho t(f_\beta) \leqslant \liminf_{n\to\infty} \left[\frac{\log^+ \{E_n \exp((\alpha-\beta)\lambda_{n+2})\}}{\lambda_{n+1}}\right]^{\rho+1}.$$

Inequolity (2.10) now follows on using (2.1). This proves Theorem 2.

Combining (2.9) and (2.10), we obtain.

Corollary—Let $f(s) \in D_\alpha$ be of order ρ, $0 < \rho < \infty$, and lower type t, $0 \leqslant t \leqslant \infty$. If $\lambda_n \sim \lambda_{n+1}$ as $n \to \infty$ and $\psi(n) = \dfrac{\log(E_{n-1}/E_n)}{\lambda_{n+1}-\lambda_n}$ forms a non decreasing function of n for all large values of n, then

$$\left(\frac{\rho+1}{\rho}\right)^{\rho+1} (\rho t) = \liminf_{n\to\infty} \lambda_n \left[\frac{\log^+ \{E_n \exp((\alpha-\beta)\lambda_{n+1})\}}{\lambda_{n+1}}\right]^{\rho+1}. \quad \ldots(2.11)$$

§3. In this section we obtain the characterizations of ρ, λ, T, and t in terms of the ratio of error terms (E_{n-1}/E_n). We prove

Theorem 3—Let $f(s) \in D_\alpha$ be of order ρ and $-\infty < \beta < \alpha < \infty$.

If $\psi(n) = \dfrac{\log(E_{n-1}/E_n)}{\lambda_{n+1}-\lambda_n}$ forms a non-decreasing function of n for all large values of n then

$$\rho + 1 = \limsup_{n\to\infty} \frac{\log \lambda_{n+1}}{\log(\lambda_{n+1}-\lambda_n) - \log^+[(\alpha-\beta)(\lambda_{n+1}-\lambda_n) + \log(E_n/E_{n-1})]}. \quad \ldots(3.1)$$

Proof : Let the expression on the right hand side of (3.1) be denoted by $B+1$. We first assume that $B < \infty$. Then for $\epsilon > 0$ and arbitrarily small,

$$\frac{\log \lambda_{n+1}}{\log(\lambda_{n+1}-\lambda_n) - \log^+[(\alpha-\beta)(\lambda_{n+1}-\lambda_n) + \log(E_n/E_{n-1})]} < B + 1 + \epsilon,$$
$$n > N(\epsilon)$$

or,

$$\log(E_n/E_{n-1}) < (\lambda_{n+1}-\lambda_n)(\lambda_{n+1})^{-1/(B+1+\epsilon)} - (\alpha-\beta)(\lambda_{n+1}-\lambda_n),\ n > N(\epsilon).$$

Substituting $n = N+1, N+2, \ldots, k$ in the above inequality and adding, we obtain

$$\log(E_k/E_N) < \sum_{n=N+1}^{k} (\lambda_{n+1}-\lambda_n)(\lambda_{n+1})^{-1/(B+1+\epsilon)} - (\alpha-\beta) \sum_{n=N+1}^{k} (\lambda_{n+1}-\lambda_n).$$

Putting $c = 1/(B+1+\epsilon)$ and rearranging the terms on the right hand side, we obtain

$$\log E_k < \lambda_{k+1}^{1-c} - \lambda_{N+1}\ \lambda_{N+2}^{-c} - \sum_{n-N+2}^{k} \lambda_n \left(\lambda_{n+1}^{-c} - \lambda_n^{-c}\right)$$

$$+ (\alpha - \beta)(\lambda_{N+1} - \lambda_{k+1}) + \log E_N.$$

Let us put $n(t) = \lambda_n$ for λ_n for $\lambda_n \leqslant t < \lambda_{n+1}$ and $F(t) - t^{-c}$.

Then

$$\log E_k < \log E_N + \lambda_{k+1}^{1-c} - \lambda_{N+1}\lambda_{N+2}^{-c} - \int_{\lambda_{N+1}}^{\lambda_{k+1}} n(t)\, dF(t) + (\alpha-\beta)(\lambda_{N+1}-\lambda_{k+1})$$

$$= O(1) + \lambda_{k-1}^{1-c} + c\int_{\lambda_{N+1}}^{\lambda} \frac{n(t)}{t}\ t^{-c}\, dt - (\alpha - \beta)\lambda_{k+1}.$$

Since $n(t)/t \leqslant 1$ we obtain

$$[\log E_k + (\alpha-\beta)\lambda_{k+1}] < O(1) + \lambda_{k+1}^{1-c} + \frac{c}{1-c}\lambda_{k+1}^{1-c}\ , k > N$$

$$= O(1) + \frac{1}{1-c}\ \lambda_{k-1}^{1-c}\ .$$

Precceeding to limits as $k \to \infty$, we obtain

$$\limsup_{k\to\infty} \frac{\log^+ \log^+ [E_k \{\exp(\alpha - \beta)\lambda_{k+1}\}]}{\log \lambda_{k+1}} \leqslant \frac{B}{B-1}$$

and in view of (1.11), we obtain $\rho/\rho+1 \leqslant B/B+1$. Hence we have $\rho \leqslant B$.

The above proof obviously holds for $B \geqslant 0$ only. However, if we assume $B < 0$ then for suitable $\epsilon > 0$, $1 - e < 0$.

Then

$$\log^+ \{E_k \exp((\alpha - \beta)\lambda_{k+1})\} < O(1)$$

and hence from (1.11) it follows that $\rho = 0$ while we have assumed that $0 < \rho < \infty$.

The inequality obviously holds if $B = \infty$.

To prove the reverse inequality, we use the condition given for $\psi(n)$. Then for all $n \geqslant n_0$ (fixed), we can write

$$\log(E_{n_0}/E_n) = \sum_{k=n_0}^{n} \psi(k)(\lambda_{k+1} - \lambda_k)$$

$$\leqslant \psi(n)(\lambda_{n+1} - \lambda_{n_0})$$

or,

$$(\alpha - \beta) + \frac{\log (E_n/E_{n-1})}{\lambda_{n+1} - \lambda_n} \leqslant (\alpha - \beta) + \frac{\log^+ E_n}{\lambda_{n+1}} + O(1), \; n > n_0. \quad \ldots(3.2)$$

Hence

$$\limsup_{n\to\infty} \frac{\log \lambda_{n+1}}{\log (\lambda_{n+1} - \lambda_n) - \log^+ [(\alpha - \beta)(\lambda_{n+1} - \lambda_n) + \log (E_n/E_{n-1})]}$$

$$\leqslant \limsup_{n\to\infty} \frac{\log \lambda_{n+1}}{\log \lambda_{n+1} - \log^+ [\log^+ E_n + (\alpha - \beta)\lambda_{n+1}]}.$$

Hence $B + 1 \leqslant \rho + 1$ in view of (1.11). This proves that $B = \rho$ and hence Theorem 3 follows.

Theorem 4—Let $f(s) \in D_\alpha$ be of order $\rho > 0$ and lower order λ, $0 \leqslant \lambda \leqslant \infty$. Then

$$1 + \lambda \geqslant \liminf_{n\to\infty} \frac{\log\lambda_n}{\log^+ [(\lambda_{n+1} - \lambda_n)/\{(\alpha - \beta)(\lambda_{n+1} - \lambda_n) + \log (E_n/E_{n-1})\}]}. \quad \ldots(3.3)$$

Further, if $\psi(n) = \dfrac{\log (E_{n-1}/E_n)}{\lambda_{n+1} - \lambda_n}$ forms a non-decreasing function of n for all large values of n and $f_\beta(s)$ belonge to the "admissible class" then equality holds in (3.3).

PROOF : Let us put the right hand side of (3.3) equal to θ. We first consider the case when $1 < \theta < \infty$. For arbitrary ϵ, $0 < \epsilon < \theta$, we have

$$\log (E_n/E_{n-1}) > (\lambda_{n+1} - \lambda_n)\{\lambda_n^{-1/(\epsilon-\theta)} - (\alpha - \beta)\}, \; n > N = N(\epsilon).$$

Proceeding as before we obtain

$$\log E_n > \log E_N + \sum_{k=N+1}^{n} (\lambda_{k+1} - \lambda_k)\, [\lambda_k^{-1/(\epsilon\theta)} - (\alpha - \beta))$$

$$> \log E_N + [\lambda^{-1/(\theta-\epsilon)} - (\alpha - \beta)](\lambda_{n+1} - \lambda_n),$$

since λ_n is an increasing sequence, Proceeding an in the proof of Theorem 1, we can prove what $1 + \lambda \geqslant \theta$. This inequality obviously holds if $0 \leqslant \theta \leqslant 1$. For $\theta = \infty$, above method gives (3.3) if we replace $(\theta - \epsilon)$ by any arbitraily large number.

To complete the proof, we use the condition on $\psi(n)$. Then from (2.8) and Corollary of Nandan[4] (p. 1367), we get for the analytic function $f_\beta(s)$ satisfying 1.9).

$$1 + \lambda(f_\beta) = \liminf_{n\to\infty} \frac{\log \lambda_n}{\log^+ [(\lambda_{n+1} - \lambda_n)/\{\alpha(\lambda_{n+1} - \lambda_n) + \log (b_{n+1}/b_n)\}]}$$

where $b_n = E_{n-1} \exp(-\beta\lambda_n)$. In view of (2.1) and (3.3) we now get $1 + \lambda = 0$. This proves Theorem 4.

Lastly we prove

Theorem 5—Let $f(s) \in D_\alpha$ be of order ρ, $0 < \rho < \infty$, type T and lower type t, $0 < t \leqslant T < \infty$.

(i) We have

$$\rho T \leqslant \theta \leqslant \left(\frac{\rho + 1}{\rho}\right)^{\rho+1} (\rho T) \qquad \ldots(3.4)$$

where the right hand inequality of (3.4) holds if $\psi(n)$ forms a non-decreasing function of n for all large n and

$$\theta = \limsup_{n\to\infty} \lambda_{n+1} \left[\frac{\log(E_n/E_{n-1}) + (\alpha - \beta)(\lambda_{n+1} - \lambda_n)}{\lambda_{n+1} - \lambda_n}\right]^{\rho+1}.$$

(ii) If $\lambda_n \; \lambda_{n+1}$ as $n \to \infty$, then

$$\rho t \geqslant \liminf_{n\to\infty} \lambda_n \left[\frac{\log(E_n/E_{n-1}) + (\alpha - \beta)(\lambda_{n-1} - \lambda_n)}{\lambda_{n+1} - \lambda_n}\right]^{\lambda+1}. \qquad \ldots(3.5)$$

PROOF: The right-hand inequality of (3.4) follows immediately on using (3.2) and (1.12) of Theorem *B*. The proof of left-hand inequality of (3.4) i. e. $\rho T \leqslant \theta$ follows on the lines of proof of first half of Theorem 3. Hence we omit the proof.

To prove (3.5), we denote the right-hand side expression of (3.5) by q. Then for arbitrary ϵ, $0 < \epsilon < q$, we have

$$\log(E_n/E_{n-1}) + (\alpha - \beta)(\lambda_{n+1} - \lambda_n) > (\lambda_{n+1} - \lambda_n)\left(\frac{q - \epsilon}{\lambda}\right)^{1/(\rho+1)}$$

$$n > N(\epsilon).$$

Now

$$\log E_k = \sum_{n=N+1}^{k} \log(E_n/E_{n-1}) - \log E_N$$

$$> \sum_{n=N+1}^{k} (\lambda_{n+1} - \lambda_n)\left(\frac{q - \epsilon}{\lambda_n}\right)^{1/(\rho+1)} - \sum_{n=N+1}^{k} (\alpha - \beta)(\lambda_{n+1} - \lambda_n).$$

$$- \log E_N.$$

Putting $c = 1/(\rho + 1)$ and rearranging the trems on right-hand side we get

$$\log E_k > (q - \epsilon)^c \left[\lambda_{k+1} \lambda_k^{-c} - \lambda_{N+1}^{1-c} \sum_{n=N+2}^{k} \lambda_n \left(\lambda_{n-1}^{-c} - \lambda_n^{-c}\right)\right]$$

$$- (\alpha - \beta)\lambda_{k+1} + O(1), \; k > N.$$

We put $n(t) = \lambda_n$ for $\lambda_{n-1} < t \leqslant \lambda_n$, $F(t) = t^{-c}$. Then

$$\log E_k > (q-\epsilon)^c \left[\lambda_{k+1}\lambda_k^{-c} - \lambda_{N+1}^{1-c} - \int_{\lambda_{N+2}}^{\lambda_k} n(t)\, dF(t)\right] - (\alpha-\beta)\lambda_{k+1} + O(1)$$

or

$$\log E_k + (\alpha-\beta)\lambda_{k+1} > O(1) + (q-\epsilon)^c \left[\lambda_{k+1}\lambda_k^{-c} + c\int_{\lambda_{N+2}}^{\lambda_k} \frac{n(t)\,t^{-c}}{t}\, dt\right].$$

Since $n(t)/t \geqslant 1$, we have

$$\log E_k + (\alpha-\beta)\lambda_{k+1} > O(1) + (q-\epsilon)^c \left[\lambda_{k+1}\lambda_k^{-c} + \frac{c}{1-c}\left(\lambda_k^{1-c} - \lambda_{N+2}^{1-c}\right)\right] > O(1) + \frac{(q-\epsilon)^c \lambda_k^{1-c}}{1-c}, \quad k > N$$

since λ_n is an increasing sequence. Hence on proceeding to limits, we obtain

$$\liminf_{k\to\infty} \lambda_k \left[\frac{\log^+ E_k + (\alpha-\beta)\lambda_{k+1}}{\lambda_k}\right]^{\rho+1} \geqslant \left(\frac{\rho+1}{\rho}\right)^{\rho+1} q. \qquad \ldots(3.6)$$

Since $\lambda_n \sim \lambda_{n+1}$ as $n \to \infty$, combining (3.6) and (2.9) we get (3.5). This completes the proof of Theorem 5.

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ON SPATIAL NUMERICAL RANGES OF OPERATORS ON BANACH SPACES

C. PUTTAMADAIAH AND HUCHE GOWDA

Department of Mathematics, University of Mysore, Manasagangotri Mysore 570006

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In this paper a characterization for the spatial numerical range of a normal operator on a smooth reflexive Banach space, to be closed and convex, is given (Theorem 1). This generalizes the Theorem of Ching-Hua-Meng[3], for normal operators on Hilbert spaces. A few more results concerning the spatial numerical ranges of convexoid and isoabelian operators are also obtained.

Let X be a complex Banach space and $B(X)$ the Banach algebra of bounded linear operators on X. For $T \in B(X)$, the spatial numerical range $V(T)$ and the numerical range $W(T)$ of T are given by

$$V(T) = \{f(Tx) : x \in X, f \in X^*, \|f\| = \|x\| = f(x) = 1\}$$

and

$$W(T) = \{[Tx, x] : \|x\| = 1\}$$

where $[.,.]$ is a consistent semi-inner-product (s.i.p) of Lumer[8]. The sets $V(T)$ and $W(T)$ are neither closed nor convex in general. It is easy to see that $V(T)$ is the union of all the numerical ranges $W(T)$ corresponding to all consistent s.i.p's on X. However, $V(T) = W(T)$ in case of smooth space.

$T \in B(X)$ is said to be hermitian if $W(T)$ is real or equivalently $V(T)$ is real. If $T = H + iK$, where H and K are hermitian with $HK = KH$, then T is called a normal operator. The numerical range of T w.r.t the Banach algebra $B(X)$ is defined by

$$V(B(X), T) = \{F(T) : F \in B(X)^*, \|F\| = F(I) = 1\}.$$

It is easy to see that $W(T) \subset V(T) \subset V(B(X), T)$.

An extreme point of a convex set $S \subset \mathbb{C}$ is a point μ of S which does not belong to any line segment joining two points of S which are different from μ.

The sets $\sigma(T)$, $Co\,\sigma(T)$, $\sigma_c(T)$, $\sigma_p(T)$, $\sigma_r(T)$, $\sigma_\pi(T)$ and $\Gamma(T)$ respectively denote the spectrum, convex hull of the spectrum, continuous spectrum, point spectrum, residual

spectrum, approximate point spectrum and the compression spectrum of T. An operator T is called convexoid if $\overline{V(T)} = Co\ \sigma(T)$. A normal operator on a Banach space X is convexoid. Indeed, if T is a normal operator, then $V(B(X), T) = Co\ \sigma(T)$ (Bonsall and Duncan[1], Th. 14, p. 54). Since $Co\ \sigma(T) \subset \overline{V(T)}$ for any $T \in B(X)$ (Bonsall and Duncan[2], Th. 4, p. 22) and $\overline{V(T)} \subset V(B(X), T)$, we have $\overline{V(T)} = Co\ \sigma(T)$.

Throughout this this E denotes the set of all extreme points of $\overline{V(T)}$ of a convexoid operator T.

Definition— A Banach space X is said to be smooth if for each unit vector x in X, the set $D(x) = \{f \in X^* : \|f\| = f(x) = 1\}$ contains only one point.

Theorem 1—Let T be a normal operator on a smooth reflexive Banach space X. Then $V(T)$ is closed and convex if and only if $E \cap \sigma_c(T) = \phi$.

Proof : Since T is normal, $\overline{V(T)} = Co\ \sigma(T)$, and therefore $E \subset \sigma(T)$. Suppose $E \cap \sigma_c(T) = \phi$. Then as $\sigma_r(T) = \phi$ (Mattila[9], Th. 4.7), $E \subset \sigma_p(T)$. Since $\overline{V(T)}$ is a compact convex subset of C, we observe that $\overline{V(T)} = Co\ E$. As $Co\ \sigma_p(T) \subset V(T)$ (Bonsall and Duncan[2], Th. 3, p. 21), it follows that $\overline{V(T)} \subset V(T)$. Thus $V(T)$ is closed and convex.

Conversely, suppose $V(T)$ is closed and convex. Since $V(T) \subset V(T^*) \subset \overline{V(T)}$ (Bonsall and Duncan[2], Corollary 3, p. 12), $V(T) = V(T^*)$. T^* is normal because T is normal. As X is smooth and reflexive, X^* is strictly convex, and therefore

$$E = E \cap V(T) = E \cap V(T^*) \subset \sigma_p(T^*) \text{ (Mattila[9], Th. 7.2).}$$

Now the reflexivity of X and X^* impliles that $\sigma_c(T) = \sigma_c(T^*)$ (Goldberg[5], p. 71) and $\sigma_r(T) = \phi = \sigma_r(T^*)$ (Mattila[9], Th. 4.7). Therefore, $\sigma_p(T) = \sigma_p(T^*)$, and hence $E \subset \sigma_p(T)$. Thus $E \cap \sigma_c(T) = \phi$.

We note that Theorem 1 is an extension of the Theorem proved by Ching-Hua-Meng[3] for normal operators on a Hilbert space.

Proposition 2—Let T be a normal operator on a separable reflexive Banach space X. If $E \cap \sigma_c(T) = \phi$, then E is countable.

Proof : By the first part of Theorem 1, $E \subset \sigma_p(T)$. Since X is separable, $\sigma_p(T)$ E is countable by (Mattila[9], Cor. 3.10).

Proposition 3—Let T be a hermitian operator on a reflexive Banach space. If $E \cap \sigma_c(T) = \phi$, then there exists an eigenvalve α such that $|\alpha| = \|T\|$.

Proof : Since $\overline{V(T)} = Co\ \sigma(T)$, $E \subset \sigma(T)$. As T is normaloid (Sinclair[12], Prop. 2) there is an α in $\sigma(T)$ such that $|\alpha| = \|T\|$, and hence α is in E. By the first part of Theorem 1, $E \subset \sigma_p(T)$.

Theorem 4—Let T be a convexoid operator on a Banach space. Then

(i) $E \subset \sigma_\pi(T)$

(ii) If $E \subset \sigma_p(T)$, $V(T)$ is closed and convex.

PROOF : (i) Since $\overline{V(T)} = Co\,\sigma(T)$, $E \subset \sigma(T)$. As $\sigma(T) \subset \overline{V(T)}$, E canot be in the interior of $\sigma(T)$. Hence $E \subset \partial\sigma(T) \subset \sigma_\pi(T)$.

(ii) Since $Co\,\sigma(T) = \overline{V(T)} = Co\,E$ and $Co\,\sigma_p(T) \subset V(T)$ (Bonsall and Duncan[2], Th. 3), it follows that $V(T)$ is closed and convex.

Remark 1 : $V(T) \neq V(T^*)$ in general. However they are equal for a convexoid operator T on a Banach space with $E \subset \sigma_p(T)$. This follows from (ii) of Theorem 4 and Corollary 3 of Bonsall and Duncan[2] (p. 12).

Remark 2 : It would be interesting to know whether the converse of (ii) in Theorem 4 is true. However, we prove

Theorem 5—Let T be a normal operator on a strictly convex Banach space. Then a necessary and sufficient condition that $V(T)$ be closed and convex is that $E \subset \sigma_p(T)$.

PROOF : Since a normal operator is convexoid, the sufficiency follows from (ii) of Theorem 4. Also the condition is necessary for, if $V(T)$ is closed, then

$E = E \cap V(T) \subset \sigma_p(T)$ by (Mattila[9], Th. 7.2).

A non zero vector x in a normed linear space X is said to be orthogonal to y in X w.r.t a compatible s.i.p. [.,.] on X if $[y, x] = 0$. This orthogonality is not symmetric in general.

Lemma 6—If M is a proper closed subspace of a reflexive Banach space X, then there is a non-zero vector y in X and a compatible s.i.p [.,.] on X such that y is orthogonal to M.

This is our Corollary 9 of Puttamadaiah and Huchegowda[11].

Theorem 7—If X is a reflexive Banach space and $T \in B(X)$, then $\Gamma(T) \subset V(T)$; in particular $\sigma_r(T) \subset V(T)$.

PROOF : Let $\mu \in \Gamma(T)$. Then $(T - \mu I)X$ is not dense in X. Then by Lemma 6, there is a compatible s i.p [.,.] on X and a non zero vector z in X such that $[(T - \mu I)x, z] = 0$ for all x in X. In particular, for $x = z$, $\mu = [Tz, z]$ with $\|z\| = 1$ and hence $\mu \in W(T) \subset V(T)$.

Mattila[9] (Th. 7.3) has proved that if T is a normal operator on a smooth reflexive Banach space, then $E \cap V(T) \subset \sigma_p(T)$; consequently $E - V(T) \subset \sigma_c(T)$ because $\sigma_r(T) = \phi$ (Mattila[9], Th. 4.7). We now extend the latter result for convexoid operators on a reflexive Banach space.

Theorem 8—If T is a convexoid operator on a reflexive Banach space X, then $E - V(T) \subset \sigma_c(T)$.

PROOF : Since $\overline{V(T)} = Co\, \sigma(T)$, $E \subset \sigma(T)$. As $\sigma_p(T) \subset V(T)$ always and $\sigma_r(T) \subset V(T)$ by Theorem 7, it follows that

$$E - V(T) \subset \sigma(T) - V(T) \subset \sigma_c(T).$$

For each x in a normed linear space X, there is an x^* in X^* such that $x^*(x) = \|x\|^2$ and $\|x^*\| = \|x\|$. Let ϕ associate each x in X to exactly one such x^* in X^* and αx to $\bar{\alpha}x^*$. ϕ is called a support mapping. There are infinite number of such mappings unless the space is smooth. Every support mapping ϕ defines a compatible s.i.p $[.,.]$ on X if we set $\phi(x)(y) = [y, x]$. An invertible operator T on a Banach space X is said to be is *so*-abelian if there is a support mapping ϕ such that $\phi T = T^{*-1}\phi$, or equivalently $(Tx)^* = T^{*-1} x^*$ or $[Tx, y] = [x, T^{-1} y]$ for all x, y in X where $[.,.]$ is a consistent s.i.p on X defined by $\phi(x)(y) = [y, x]$.

We note that the following three conditions are equivalent for a bounded linear operator T on a Banach space.

(1) T is iso-abelian

(2) T is an invertible isometry

(3) T is invertible and $\|T\| = \|T^{-1}\| = 1$.

(1) $\Leftrightarrow$ (2) (Koehler and Rosenthal[6], Cor. 1). It is easy to see the equivalence of (2) and (3).

Theorem 9—If X is a reflexive Banach space and $T \in B(X)$ with $0 \notin V(T)$, then

(i) T is $1 - 1$

(ii) $\overline{TX} = X$

(iii) If T is an isometry on X, then T is iso-abelian.

PROOF : (i) Suppose x is a unit vector such that $Tx = 0$. Then by Hahn Banach theorem there is an $f \in X^*$ such that $\|f\| = 1$ and $f(x) = 1$. Now $0 = f(Tx) \in V(T)$, a contradiction. We note that X need not be reflexive in this case.

(ii) Suppose $\overline{TX} \neq X$. Then by Lemma 6, there is a compatible s.i.p $[.,.]$ on X and a non zero vector y such that $[Tx, y] = 0$ for all x in X. In particular, $[Ty, y] = 0$. Since we can take $\|y\| = 1$, $0 \in W(T) \subset V(T)$, this contradiction proves (ii).

(iii) Since T is an isometry, TX is closed, and hence by (ii) T is onto and so it is an invertible isometry. Thus T is iso-abelian.

A point $\mu \in \partial\sigma(T)$ is said to be a proper boundary point of $\sigma(T)$ if there exists a bounded sequence (μ_n) in $\rho(T)$ such that $\|(\mu_n - \mu)(\mu_n - T)^{-1}\| \to 1$ as $n \to \infty$, where $\rho(T)$ is the resolvent set of T. The set of all proper boundary points of $\sigma(T)$ is denoted by $p_r(T)$.

We observe that if T is iso-abelian, then so is T^*.

Theorem 10—If T is an iso-abelian operator on a reflexive Banach space X, then $\sigma_r(T) = \phi$.

PROOF : If $\mu \in \sigma(T)$, then $|\mu| = \|T\|$ and so $\mu \in \partial V(B(X), T)$. Since $\sigma(T) \cap \partial V(B(X), T) \subset p_r(T)$ (Mattila[10], Lemma 1), $\mu \in P_r(T)$ or equivalently $0 \in P_r(T - \mu I)$. Since X is reflexive, $X = \text{Ker}(T - \mu I) \oplus \overline{(T - \mu I)X}$ by (Mattila[9], Cor. 4.5). This shows that $\sigma_r(T) = \phi$.

Corollary 11—If T is an iso-abelian operator on a reflexive Banach space X, then $\sigma_p(T) = \sigma_p(T^*)$.

PROOF : Since T^* is iso-abelian, the reflexivity of X and X^* implies that $\sigma_c(T) = \sigma_c(T^*)$ (Goldberg[5], p. 71) and $\sigma_r(T) = \phi = \sigma_r(T^*)$ by Theorem 10.

Theorem 12—If T is an iso-abelian operator on a smooth reflexive Banach space X, then $\sigma(T) \cap V(T) \subset \sigma_p(T)$.

PROOF : It is clear that $\sigma(T^*)$ lies on the unit circle. Since X is smooth and reflexive, X^* is strictly convex, and therefore by (Bonsall and Duncan[1], Th. 8, p. 93) $\sigma(T^*) \cap V(T^*) \subset \sigma_p(T^*)$. Since $V(T) \subset V(T^*)$ and $\sigma(T) = \sigma(T^*)$, $\sigma(T) \cap V(T) \subset \sigma(T^*) \cap V(T^*)$. The proof is completed by Corollary 11.

Corollary 13—If T is an iso-abelian operator on a smooth reflexive Banach space X, then the extreme points of $Co\,\sigma(T)$ which lie in $V(T)$ are the eigen values of T.

Theorem 14—If T is a normal and iso-abelian operator on a smooth reflexive space X, then $V(T)$ is closed if and only if the spectrum of T consists entirely of the point spectrum.

PROOF : Since T is an invertible isometry, $\sigma(T)$ lies on the unit circle. As T is normal, $\overline{V(T)} = Co\,\sigma(T)$ and therefore it follows that $E = \sigma(T)$. Suppose $V(T)$ is closed. Then by Theorem 1, $E \cap \sigma_c(T) = \phi$. But this holds only if $\sigma(T) = \sigma_p(T)$ because $\sigma_r(T) = \phi$, by Theorem 10.

The converse is an immediate consequence of the fact that T is convexoid and $Co\,\sigma_p(T) \subset V(T)$ (Bonsall and Duncan[2], Th. 3, p. 21).

Theorem 15—If T is an iso-abelian (or normal) operator on a Banach space with descent $\partial(T - \mu I)$ finite for each μ in $\sigma(T)$ then $Co\,\sigma(T) \subset V(T)$.

PROOF : By Corollary 3.6 of Mattila[9], the ascent $\alpha(T - \mu I) \leqslant 1$ for each μ in C and hence $\partial(T - \mu I) = \alpha(T - \mu I) \leqslant 1$ for each μ in $\sigma(T)$. Now μ is an isolated

point of $\sigma(T)$ and is a pole of the resolvent of T (Lay[7], Th. 2.1). Therefore μ is an eigen value of T (Taylor[13], Th. 5.8 — A). Hence $\sigma(T) = \sigma_p(T)$. Since $Co\, \sigma_p(T) \subset V(T)$ (Bonsall and Duncan[2], Th. 3), the proof is completed.

Corollary 16—Under the hypothesis of the Theorem, $\sigma(T) \subset W(T)$.

Corollary 17—If T is a normal operator on a Banach space with $\partial(T - \mu I)$ finite for each μ in $\sigma(T)$, then $V(T)$ is closed and convex.

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ON MULTIPLIERS FOR THE ABSOLUTE MATRIX SUMMABILTTY

M. L. MITTAL, G. PRASAD* AND NARENDRA KUMAR

Department of Mathematics, University of Roorkee, Roorkee 247672

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Yadav[7] proved $\phi_1(t) \in BV(0, \pi)$ implies $\Sigma A_n e_n \in | N, p_n |$ under certain conditions on $\{p_n\}$ provided $\{e_n\} \in BV(0, \infty)$. In the present note, we have extended the above result to matrix summability.

§1. Let Σu_n be a given infinite series with sequence of partial sums $\{s_n\}$. Let $\|T\| \equiv (a_{n,k})$ be an infinite triangular matrix with real constants. Then sequence-to-sequence transformation

$$t_n = \sum_{k=0}^{n} a_{n,k} s_k, \ n = 0, 1, 2, \ldots \qquad \ldots(1.1)$$

defines the T-transform of the sequence $\{s_n\}$. The series Σu_n is said to be T-summable to s, if $\lim_{n\to\infty} t_n = s$, and is said to be absolutely T-summable or simply summable $| T |$ if the infinite series $\sum_{n=1}^{\infty} | t_n - t_{n-1} |$ is convergent.

If matrix element $a_{n,k} = 0$, for every $k > n$, then the matrix is called triangular matrix. The matrix T-reduces to Nörlund matrix generated by the sequence of coefficients $\{p_n\}$, if

$$a_{n,k} = \begin{cases} \dfrac{p_{n-k}}{P_n}, & \text{if } k \leqslant n; \\ 0, & \text{if } k > n; \end{cases}$$

where

$$P_n = \sum_{k=0}^{n} p_k \neq 0.$$

§2. Let $f(t)$ be a function integrable (L) over $(-\pi, \pi)$ and periodic with period 2π. We assume, without any loss of generality that the constant terms in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)\, dt = 0$$

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and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t). \qquad \ldots(2.1)$$

We write

$$\phi(t) = \tfrac{1}{2}[f(x+t) + f(x-t)];$$

$$\Phi_0(t) = \phi(t);$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du, \ \alpha > 0;$$

$$\phi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha} \Phi_\alpha(t), \ \alpha \geqslant 0;$$

$$D_n(t) = \tfrac{1}{2} + \cos t + \cos 2t + \ldots + \cos nt = \frac{\sin(n+\frac{1}{2})t}{2 \sin t/2}$$

$$A_{n,k} = \sum_{r=k}^{n} a_{n,r};\ V_{n,k} = \frac{(n-k+1)\, a_{n,k}}{A_{n,k}}$$

$$\Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}$$

$$\tau = [1/t],$$

where $[\lambda]$ is an integral part of λ.

Furthermore, A will denote an absolute constant and we will write $A + A = A$ and $A - A = A$.

§3. Bhatt[1] proved the following :

Theorem A—Let $\{p_n\}$ be a non-negative and non-decreasing sequence of real numbers such that

$$\{p_{n+1} - p_n\} \text{ is ultimately monotonic,} \qquad \ldots(3.1)$$

$$\left\{ \frac{(n+1)\, p_n}{P_n} \right\} \in BV \qquad \ldots(3.2)$$

$$\frac{P_k}{k} \sum_{n=k}^{\infty} \frac{1}{P_n} \leqslant C \qquad \ldots(3.3)$$

for $k = 1, 2, \ldots$, where C is a fixed positive constant. If $\phi_1(t) \in BV(0, \pi)$, then the series (2.1), at $t = x$, is summable $|N, p_n|$.

The special case of this theorem in which (N, p_n) is (C, α) with $\alpha > 1$, is included in a more general result proved much earlier by Bosanquet[2].

Recently Yadav[7] generalized the above theorem as follows :

Theorem B—Let $\{p_n\}$ be non-negative and non-decreasing and satisfy (3.1), (3.2), (3.3) and $\{e_n\} \in BV(0, \infty)$, then

$$\phi_1(t) \in BV(0, \pi) \Rightarrow \sum_1^\infty e_n A_n(t) \in |N, p_n| .$$

The problem of absolute matrix summability of series (2.1) where $\phi(t) \in BV(0, \pi)$ has been studied by Kishore and Hotta[4], Varshney[6], Kuttner and Sahney[5] and Jurkat *et al.*[3]

In this note, we prove :

Theorem 1—Let $\|T\| \equiv (a_{n,k})$ be an infinite triangular matrix with $a_{n,k} \geqslant 0$; $A_{n,0} = 1$, $\forall\, n \geqslant 0$ such that

$$\{a_{n,k}\}_{k=0}^{n} \quad \text{...(3.4)}$$

is non-negative and non-increasing sequence with respect to k

$$\{a_{n,k} - a_{n,k+1}\}_{k=0}^{n-1} \quad \text{...(3.5)}$$

is either monotonic non-increasing or non-decreasing sequence with respect to k;

$$\sum_{k=1}^{r-1} \sup_{k<n\leqslant r} |\Delta_k V_{n,n-k}| = O(1) \quad \text{...(3.6)}$$

$$\sum_{n=k}^{\infty} A_{n,n-k} \leqslant C(k+1), \text{ for every } k \quad \text{...(3.7)}$$

and

$$a_{n-1,k} \geqslant (1 + a_{n,0})\, a_{n,k+1} \quad \text{...(3.8)}$$

for every $0 \leqslant k < n$, $n \geqslant 1$, where C is a fixed constant. If $\phi_1(t) \in BV(0, \pi)$, then the series $\sum_1^\infty e_n A_n(t)$, at $t = x$, is summable $|T|$, where $\{e_n\} \in BV(0, \infty)$.

We note that in the case of Nörlund summability (3.8) is automatically satisfied while condition (3.4) implies that the sequence $\{p_n\}$ is non-negative and non-decreasing. Also condition (3.5), (3.6) and (3.7) reduce to condition (3.1), (3.2) and (3.3) respectively.

We shall require the following Lemmas :

§4. *Lemma*[6] 1— Let $\|T\| \equiv (a_{n,k})$ be an infinite triangular matrix with $a_{n,k} \geqslant 0$, $A_{n,0} = 1$ and let $\{a_{n,k}\}$ also satisfy (3.6), (3.8) and

$$\sum_{n=k}^{\infty} \frac{A_{n,n-k}}{n+1} \leqslant C. \quad \text{...(4.1)}$$

If Σu_n is bounded then a necessary and sufficient condition for i to be summable $| T |$, is that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)} \left| \sum_{k=1}^{n} k\, a_{n,k}\, u_k \right| < \infty.$$

Lemma 2—If $\{b_{n,k}\}$ is non-negative and non-decreasing sequence with respect to k, then for $0 \leqslant a < d \leqslant \infty$, $0 \leqslant t \leqslant \pi$ and for every n and a

$$\left| \sum_{k=a}^{d} e_k\, b_{n,n-k}\, e^{i(n-k)t} \right| < A B_{n,n-\tau}$$

where

$$B_{n,k} = \sum_{r=k}^{n} b_{n,r} \text{ and } \{e_k\} \in BV(0, \infty).$$

PROOF : We have $e_k = \alpha_k - \beta_k$ such that $\{\alpha_k\}$ and $\{\beta_k\}$ are non-negative and non-increasing for $\{e_k\} \in BV(0, \infty)$.

Now

$$\left| \sum_{a}^{d} e_k\; b_{n,n-k}\, e^{i(n-k)t} \right|$$

$$= \left| \sum_{a}^{d} \alpha_k\, b_{n,n-k}\, e^{i(n-k)t} - \sum_{a}^{d} \beta_k\, b_{n,n-k}\, e^{i(n-k)t} \right|$$

$$\leqslant \alpha_a \max_{a \leqslant d_1 \leqslant d} \left| \sum_{a}^{d_1} b_{n,n-k}\, e^{i(n-k)t} \right|$$

$$+ \beta_a \max_{a < d_1 \leqslant d} \left| \sum_{a}^{d_1} b_{n,n-k}\, e^{i(n-k)t} \right|$$

$$\leqslant A_1 B_{n,n-\tau} + A_2 B_{n,n-\tau} = A\, B_{n,n-\tau}$$

by using a Lemma[4].

Lemma 3—If $\{a_{n,k}\}_{k=0}^{n}$ is a non-negative sequence such that $A_{n,0} = 1$ and satisfy (3.6), there exists a constant M such that, for all

$$V_{n,k} \leqslant M,\; 0 \leqslant k \leqslant n. \tag{4.2}$$

PROOF : If $a_{n,k} \geqslant 0$, $A_{n,0} = 1$ and satisfy (3.6) then for $0 \leqslant k \leqslant n$, we have

$$V_{n,n-k} \leqslant M.$$

Since

$$V_{n,n}=1,\ V_{n,n-1} = \frac{2a_{n,n-1}}{a_{n,n-1} + a_{n,n}} \leqslant 2$$

the result hold for $k = 0, 1$. If $2 \leqslant k \leqslant n$, then

$$| V_{n,n-k} - V_{n,\, n-1} | \leqslant \sum_{r=1}^{k-1} | \Delta_r V_{n,n-r} |$$

$$\leqslant \sum_{r=1}^{n-1} | \Delta_r V_{n,n-r} | = O(1)$$

in view of (3.6).

We note that, taking the special case $k = n$, we have

$$a_{n,0} \leqslant \frac{M}{n+1}. \qquad \ldots(4.3)$$

§5. *Proof of Theorem* 1—We know[1] that if $\phi_1(t) \in BV(0, \pi)$, then the series (2.1) is convergent. We observe that (3.7) implies (4.1). In fact

$$\sum_{n=k}^{\infty} \frac{A_{n,n-k}}{n+1} \leqslant \frac{1}{k+1} \sum_{n=k}^{\infty} A_{n,n-k} \leqslant C.$$

Thus by Lemma 1, it is sufficient to prove that

$$\sum_{n} \frac{|\sigma_n|}{n+1} < \infty \qquad \ldots(5.1)$$

where

$$\sigma_n = \sum_{k=1}^{n} k\, e_k\, a_{n,k}\, A_k(x).$$

Now

$$A_k(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos kt\, dt$$

$$= -\frac{2}{\pi} \int_0^{\pi} \frac{\sin kt}{k}\, d\phi_1(t) + \frac{2}{\pi} \int_0^{\pi} t \cos kt\, d\phi_1(t)$$

by integration by parts. Hence to prove (5.1), it is enough to show that the series

$$\sum_{n} \frac{1}{(n+1)} \left| \int_0^{\pi} d\phi_1(t) \sum_{k=1}^{n} e_k\, a_{n,k} \sin kt \right| \qquad \ldots(5.2)$$

and

$$\sum_{n} \frac{1}{n+1} \mid \int_{0}^{\pi} t \, d\phi_1 \, (t) \sum_{k=1}^{n} k \, e_k \, a_{n,k} \cos kt \mid \qquad \ldots(5.3)$$

are convergent. But by hypothesis

$$\int_{0}^{\pi} \mid d\phi_1 \, (t) \mid < A$$

it suffices for our purpose to show that uniformly for $0 < t \leqslant \pi$

$$I_1 = \sum_{n=1}^{\infty} \frac{1}{n+1} \mid \sum_{k=1}^{n} e_k \, a_{n,k} \sin kt \mid \leqslant C$$

and

$$I_2 = t \sum_{n=1}^{\infty} \frac{1}{n+1} \mid \sum_{k=1}^{n} k \, e_k \, a_{n,k} \cos kt \mid \leqslant C.$$

Now

$$I_1 = \sum_{n \leqslant \tau} + \sum_{n > \tau}$$

$$= \sum_{n \leqslant \tau} \frac{1}{n+1} \mid \sum_{k=0}^{n} e_k \, a_{n,k} \sin kt \mid$$

$$+ \sum_{n > \tau} \frac{1}{n+1} \mid \sum_{k=0}^{n} e_k \, a_{n,k} \sin kt \mid$$

$$\leqslant \sum_{n \leqslant \tau} \frac{1}{n+1} \, nt \sum_{k=0}^{n} a_{n,k}$$

$$+ A \sum_{n > \tau} \frac{1}{n+1} \, a_{n,0} \max_{0 \leqslant r \leqslant n} \mid \sum_{k=0}^{r} \sin kt \mid$$

$$= t \sum_{n \leqslant \tau} A_{n,0} + O(\tau) \sum_{n > \tau} \frac{1}{(n+1)^2}$$

$$= O(1)$$

in view of (3.6), (4.3) and Abel's Lemma.

This completes the proof of (5.2).

We now proceed to prove (5.3). Using Abel's transformation, we have

$$I_2 = t \sum_{n=1}^{\infty} \frac{1}{n+1} \ | \sum_{k=1}^{n} k \, e_k \, a_{n,k} \cos kt \ |$$

$$= t \sum_{n=1}^{\infty} \frac{1}{n+1} \ | \sum_{k=1}^{n} (\Delta_k \, e_k \, a_{n,k}) \ (\sum_{r=1}^{k} r \cos rt) \ |$$

$$= \sum_{n \leqslant \tau} + \sum_{n > \tau}$$

$$= S_1(t) + S_2(t), \text{ say.}$$

Using (3.4), we observe that

$$S_1(t) \ t \leqslant \sum_{n=1}^{\tau} \frac{n^2}{(n+1)} \sum_{k=1}^{n} \ | \ \Delta_k \, (e_k \, a_{n,k}) \ |$$

$$\leqslant t \sum_{n=1}^{\tau} n \sum_{k=1}^{n} \ | \ (e_k \, \Delta_k \, a_{n,k} + a_{n,k+1} \, \Delta \, e_k) \ |$$

$$\leqslant t \sum_{n=1}^{\tau} n \sum_{k=1}^{n} \ | \ e_k \, \Delta_k \, a_{n,k} \ |$$

$$+ \ t \sum_{1}^{\tau} n \sum_{k=1}^{n} \ | \ a_{n,k+1} \, \Delta \, e_k \ |$$

$$\leqslant At \sum_{n=1}^{\tau} n \sum_{k=1}^{n} \ | \ \Delta_k \, a_{n,k} \ | + t \sum_{1}^{\tau} n \, a_{n,2} \sum_{k=1}^{n} \ | \ \Delta \, e_k \ |$$

$$\leqslant At \sum_{n=1}^{\tau} n \sum_{k=0}^{n} (a_{n,k} - a_{n,k+1}) + At \sum_{1}^{\tau} n \, a_{n,0}$$

$$\leqslant At \sum_{1}^{\tau} n \, a_{n,0}$$

$$= O(1). \qquad \ldots(5.4)$$

Now since

$$\sum_{r=1}^{n} r \cos rt = O(t^{-2}) + n \, D_n(t), \ 0 < t \leqslant \pi$$

we have

$$S_2(t) = t \sum_{n > \tau} \frac{1}{(n+1)} \ | \sum_{k=1}^{n} [\Delta_k \, (e_k \, a_{n,k})] \ (\sum_{r=1}^{k} r \text{ cor } rt) \ |$$

$$= O\,[\tau \sum_{n>\tau} \frac{1}{(n+1)} \sum_{k=1}^{n} |\,\Delta_k\,(e_k\,a_{n,k})\,|$$

$$+\; t \sum_{n>\tau} \frac{1}{(n+1)}\; |\,\sum_{1}^{n} \Delta_k\,(e_k\,a_{n,k})\,k\,D_k\,(t)\,|$$

$$= O\,[\tau \sum_{n>\tau} \frac{1}{(n+1)} \sum_{k=1}^{n} |\,\Delta_k\,(e_k\,a_{n,k})\,|$$

$$+\; t \sum_{n>\tau} \frac{1}{(n+1)}\; |\,\sum_{1}^{n} e_k\,(\Delta_k\,a_{n,k})\,k\,D_k\,(t)\,|$$

$$+\; t \sum_{n>\tau} \frac{1}{(n+1)}\; |\,\sum_{1}^{n} a_{n,k+1}\,(\Delta e_k)\,k\,D_k\,(t)\,|\,]$$

$$= K_1 + K_2 + K_3, \text{ say.} \qquad \ldots(5.5)$$

Using (3.6), we get

$$K_1 = O\,[\tau \sum_{n>\tau} \frac{1}{(n+1)} \sum_{k=1}^{n} |\,\Delta_k(e_k a_{n,k})\,|\,]$$

$$= O\,[\tau \sum_{n>\tau} \frac{1}{(n+1)}\; a_{n,0}]$$

$$= O\,[\tau \sum_{n>\tau} \frac{1}{(n+1)^2}]$$

$$= O\,(1) \qquad \ldots(5.6)$$

as in $S_1\,(t)$. In order to deal with K_2, we consider two cases separately.

Case I—Let $\{\Delta_k\,a_{n,k}\}_{k=0}^{n-1} \equiv \{b_{n,k}\}_{k=0}^{n-1}$ be monotonic non-decreasing with respect to k. Then we have by Lemma 2, and Abel's Lemma

$$|\,\sum_{r=1}^{n} e_r\,(\Delta_r\,a_{n,r})\,r\,D_r\,(t)\,|$$

$$= |\,\sum_{r=1}^{n} e_r\,b_{n,r}\;r\,D_r\,(t)\,|$$

$$\leqslant \Big| \sum_{r=1}^{n-1} e_r\, b_{n,r} \frac{r \sin (r + 1/2)t}{\sin t/2} \Big| + | ne_n\, a_{n,n}\, D_n (t)$$

$$= O\left(\frac{n}{t}\right) \max_{1\leqslant k\leqslant n-1} \Big| \sum_{r=1}^{k} b_{n,r} \sin (r + 1/2)\, t \Big|$$

$$+ O \left(\frac{n}{t} a_{n,n} | e_n | \right)$$

$$= O (n/t) [B_{n,n-\tau} + a_{n,n}]$$

$$= O (n/t) (a_{n,n-\tau} + a_{n,n}).$$

Thus

$$K_2 = O \Big[t \sum_{n>\tau} \left(\frac{n\, a_{n,n-\tau}}{t (n + 1)} \right) + a_{n,n})\Big]$$

$$= O \Big[\sum_{n>\tau} (a_{n,n-\tau} + a_{n,n})\Big]$$

$$= O \Big[\sum_{n>\tau} \frac{A_{n,n-\tau}}{\tau} + \sum_{n>\tau} A_{n,n}\Big]$$

$$= O (1) \qquad \text{...(5.7)}$$

in view of (3.7) and (4.2).

Case II —Let $\{\Delta_k (a_{n,k})\}_{k=0}^{n-1} \equiv \{b_{n,k}\}_{k=0}^{n-1}$ be monotonic non-increasing with respect to k. Now since $\sum_{r=1}^{n} r\, D_r (t) = O (n/t^2)$, so we see that inner sum in K_2 is

$$= O [n\, \tau^2 \, (a_{n,0} - a_{n,1})].$$

Hence

$$K_2 = O \Big[\tau \sum_{n>\tau} (a_{n,0} - a_{n,1})\Big].$$

But

$$a_{n,0} - a_{n,1} = \frac{1}{n + 1} \Big[(n + 1)\, a_{n,0} - \frac{na_{n,1}}{A_{n,1}}\Big] + \frac{na_{n,1}}{n + 1} \Big[\frac{1}{A_{n,1}} - 1\Big]$$

$$- \frac{a_{n,1}}{n + 1}$$

$$= \frac{1}{n + 1} \Big[(n + 1)\, a_{n,0} - \frac{na_{n,1}}{A_{n,1}}\Big] + \frac{na_{n,1}}{(n + 1)A_{n,1}} a_{n,0} - \frac{a_{n,1}}{n + 1}$$

$$= \frac{1}{(n + 1)} \Big[(n + 1)\, a_{n,0} - \frac{na_{n,1}}{A_{n,1}}\Big] + O \left(\frac{1}{n (n + 1)} \right).$$

Thus using (3.6), we have

$$K_2 = O\,[\tau \sum_{n>\tau} \frac{1}{(n+1)} \mid (n+1)a_{n,0} - \frac{na_{n,1}}{A_{n,1}} \mid] + O\,[\tau \sum_{n>\tau} \frac{1}{n(n+1)}]$$

$$= O\,(1) \qquad \ldots(5.8)$$

so that $K_2 = O\,(1)$.

Also

$$K_3 = t \sum_{n>\tau}^{n} \frac{1}{n+1} \sum_{k=1}^{n} \mid (\Delta_k\, e_k)\, a_{n,k+1}\, k\, D_k\,(t) \mid$$

$$\leqslant \sum_{n>\tau} \frac{1}{n+1} \sum_{k=1}^{n} k \mid \Delta_k\, e_k \mid a_{n,k+1}$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=1}^{n} k \mid \Delta_k\, e_k \mid a_{n,k+1}$$

$$= O\,[\sum_{k=1}^{\infty} k \mid \Delta_k\, e_k \mid \sum_{n=k+1}^{\infty} \frac{a_{n,k+1}}{n+1}].$$

But

$$\sum_{n=k+1}^{2k} \frac{a_{n,k+1}}{n+1} \leqslant \frac{}{k+2} \sum_{n=k+1}^{2k} a_{n,k+1} \leqslant \frac{C}{k+2}$$

in view of (3.4) and $A_{n,0} = 1$. We have, by Lemma 3

$$V_{n,k} = \frac{(n-k+1)\, a_{n,k}}{A_{n,k}} \leqslant M$$

i. e.

$$a_{n,k} \leqslant \frac{M\ A_{n,k}}{n-k+1} \leqslant \frac{M}{n-k+1}$$

and so

$$\sum_{n=2k+1}^{\infty} \frac{a_{n,k+1}}{n+1} \leqslant M \sum_{n=2k+1}^{\infty} \frac{1}{(n-k)\,(n+1)} = O\,\frac{1}{(k+1)}$$

i. e.

$$\sum_{n=k+1}^{\infty} \frac{a_{n,k+1}}{n+1} = O\left(\frac{1}{k+1} \right).$$

Thus, we have

$$K_3 = O\left(\sum_{1}^{\infty} |\Delta e_k|\right) = O(1). \qquad \text{...(5.9)}$$

Combining (5.4) to (5.9), we get

$$I_2 = O(1).$$

This completes the proof of Theorem 1.

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SCATTERING OF COMPRSSIONAL WAVES BY A CIRCULAR CYLINDER

P. KESARI AND B. K. RAJHANS

Department of Applied Sciences, Indian School of Mines, Dhanbad 826004

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This paper deals with the scattering of compressional waves by a circular cylinder. The cylinder is embedded in an unbounded isotropic homogeneous elastic medium and it is filled with some acoustic fluid.T he line source, generating the incident pulse is situated outside the cylinder parallel to its axis. The problem is investigated by the method of dual integral transformations. The resulting integrals are evaluated asymptotically to obtain the short time estimate of the motion near the wave front in the illuminated region of the elastic medium. We interpret the approximate solution in terms of geometrical optics.

1. INTRODUCTION

Lapwood[4] considered the problem on the disturbance due to a line source in a semi-infinite elastic medium and obtained the exact formal solution of the problem. Jeffreys and Lapwood[10] discussed the reflection of a pulse within a sphere and obtained the solution using operational methods. The problem on scattering of two dimensional elastic waves with a cylindrical obstacle in unbounded medium has been considered by various autors in recent years. Gilbert and Knopoff[7] used Friedlander's[5] method to investigate the problem of scattering of impulsive elastic waves by a rigid circular cylinder. Gilbert[6] discussed the problem of scattering of impulsive elastic waves by a smooth convex cylinder using the same method. Mishra[15,16] applied Friedlander's method to investigate the problem of scattering and diffraction of two dimensional sound pulses by a acoustically semi-transparent circular cylinder. Hwang *et al.*[13] applied a similar method and discussed the case of three dimensional elastic waves scattring by a rigid cylinder in an elastic medium.

In this paper, we investigate the short-time approximation for the scattering of compressional waves by a circular cylinder filled with inviscid fluid material. The cylinder is supposed to be situated in an unbounded homogeneous isotropic elastic medium and the incident pulse is generated by a line source situated in the surrounding elastic medium at a finite distance parallel to the axis of cylinder. We assume that the velocities of *P* and *SV* waves outside the cylinder are α and β respectively and that of *P*-waves inside the cylinder is α_0. To be specific, we suppose $\alpha > \alpha_0 > \beta$. This assumption of the velocity distribution corresponds to the actual velocity distri-

bution of elastic waves inside the earth and to the location of the source in the mantle and the outer core as the obstacle[12]. We also suppose the density of the medium outside the cylinder is ρ and that inside the cylinder is ρ' where $\rho > \rho'$. The present discussion therefore may have some relevance in seismological problems.

2. Formulation of the Problem and Formal Solution

Let the axis of the cylinder be taken as the z-axis and let a co-ordinate (r, θ) be located in the (x, y) plane with $\theta = 0$, $r = r_0$, $(r_0 > a)$ corresponding to the location of the line source which is parallel to the axis of the cylinder. The equation of the cylinder is $r = a$.

The governing equations for the present case are

$$\frac{1}{\alpha^2}\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2\varphi = \frac{2\pi}{r}\,\delta(r - r_0)\,\delta(t)\,\delta(\theta),\ (r \geqslant a) \qquad \ldots(2.1)$$

$$\frac{1}{\beta^2}\frac{\partial^2 \psi}{\partial t^2} - \nabla^2\psi = 0 \qquad (r \geqslant a) \qquad \ldots(2.2)$$

$$\frac{1}{\alpha_0^2}\frac{\partial^2 \varphi_0}{\partial t^2} - \nabla^2 \varphi_0 = 0, \qquad (r \leqslant a) \qquad \ldots(2.3)$$

where, ∇^2 is the Laplacian operator.

Besides the potentials also satisfy the conditions of quiescence at $t = 0$. The boundary conditions for the problem are that the tangential and normal components of stress vanish outside the obstacle and the radial displacement is continuous at the boundary of the obstacle[17].

Applying Laplace and Fourier transformations[5] to (2.1), (2.2) and (2.3), and following Mishra[15,16], we find that Laplace transforms of the solutions are given by

$$\bar{\varphi}(r, \theta, s) = \int_{-\infty}^{\infty} I_\mu\left(\frac{sr}{\alpha}\right) K_\mu\left(\frac{sr_0}{\alpha}\right) \exp(i\mu\theta)\, d\mu$$
$$+ \int_{-\infty}^{\infty} K_\mu\left(\frac{sr}{\alpha}\right) K_\mu\left(\frac{sr_0}{\alpha}\right)\frac{L}{M} \exp(i\mu\theta)\, d\mu,\ (r_0 \geqslant r > a) \qquad \ldots(2.4)$$

$$\bar{\psi}(r, \theta, s) = \int_{-\infty}^{\infty} K_\mu\left(\frac{sr}{\beta}\right) K_\mu\left(\frac{sr_0}{\alpha}\right)\frac{N}{M} \exp(i\mu\theta)\, d\mu,\ (r \geqslant a) \qquad \ldots(2.5)$$

and

$$\bar{\varphi}_0(r, \theta, s) = \int_{-\infty}^{\infty} \frac{L.R + N.P}{M.Q}\, I_\mu\left(\frac{sr}{\alpha_0}\right) K_\mu\left(\frac{sr_0}{\alpha}\right) \exp(i\theta\mu)\, d\mu,\ (r \leqslant a). \qquad \ldots(2.6)$$

Here L, M, N, P, Q and R are the values of the constants which are determined with the help of boundary conditions.

We see that (2.4), (2.5) and (2.6) give the integral representation of Laplace transform of the formal solutions. The time solution can be obtained on performing Laplace inversion.

3. Incident, Reflected and Refracted Pulses

We first give a brief description of the geometry of the problem. Initially the incident P-pulse striking the outer surface of the cylinder gives rise to reflected P, reflected S and refracted P-pulses, acording to the laws of ordinary geometrical optics (Fig. 1) when the rays strike the outer surface of the cylinder at critical angle, the reflected P-rays become tangential to the surface and as such they move along the surface. These surface waves at each point of their path give rise to SV-wave at

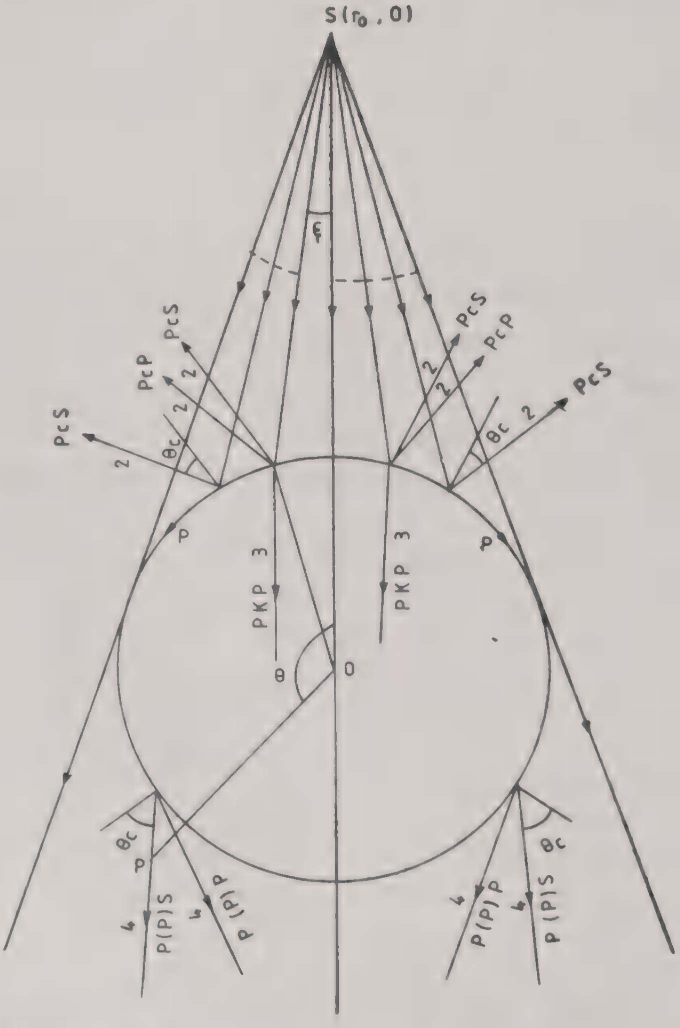

Fig. 1. (1) Incident rays, (2) reflected rays, (3) refracted rays and (4) diffracted rays.

critical angle in the outer medium and P-wave along the tangent to the surface[11]. The former are denoted by $P\,(P)\,S$ and latter by $P\,(P)\,P$ (Gilbert and Knopoff[7]). In the case of grazing incidence, the disturbances move along the surface and at each point of their path they shed diffracted P-waves in the outer medium trangential to the surface. These are denoted by $P\,(P)\,P$ (Bullen[12]). The reflected pulses are denoted by $P_c\,P$ and P_cS and refracted pulse is denoted by PKP (Bullen[12]).

Now, we use the saddle point method[9] to obtain the solution in the illuminated region of the elastic medium. Therefore using the various approximations for modified Bessel functions as given by Mishra[15,16] to the integral (2.4), (2.5) and (2.6) we find

$$\bar{\varphi}\,(r,\theta,s) \propto \left(\frac{\alpha\pi}{2\,sR_1}\right)^{1/2} \exp(-st_1) + \left(\frac{\pi}{s}\right)^{1/2} A_1\,B_1 \exp(-st_2) \qquad (r_0 \geqslant r \geqslant a) \qquad \ldots(3.1)$$

$$\bar{\psi}\,(r,\theta,s) \propto \left(\frac{\pi}{s}\right)^{1/2} A_2\,B_2 \exp(-st_3), \qquad (r \geqslant a) \qquad \ldots(3.2)$$

and

$$\bar{\varphi}_0\,(r,\theta,s) \propto \left(\frac{\pi}{s}\right)^{1/2} A_3\,B_3 \exp(-st_4), \qquad (r \leqslant a) \qquad \ldots(3.3)$$

where, t_1, t_2, t_3 and t_4 are respectively the arrival times of the incident, reflected and refracted pulses.

$$A_1 = \frac{\dfrac{4\rho\beta^3}{\alpha^2}\sin^2\eta\cos\eta\,(1-n_1^2\sin^2 n)^{1/2}(1-n^2\sin^2\eta)^{1/2} + \alpha_0\,\rho'\cos\eta - \rho\alpha\,(1-2n^2\sin^2\eta)^2(1-n_1^2\sin^2\eta)^{1/2}}{4\rho\,\beta^2\alpha^{-2}\sin^2\eta\cos\eta\,(1-n_1^2\sin^2 n)^{1/2}(1-n^2\sin^2\eta)^{1/2} + \rho'\,\alpha_0\cos\eta + \rho\alpha\,(1-2\,n^2\sin^2\eta)^2(1-n_1^2\sin^2 n)^{1/2}}$$

$$B_1 = \left[\frac{a\alpha\cos\eta}{r_0\,R_3\cos\xi + r\,R_2\cos\zeta}\right]^{1/2}$$

A_2, B_2 and A_3, B_3 have similar expressions as above.

R_1, R_2, R_3 denote distances between the source and receiver and the angles are defined in Figs. 2, 3, 4 and 5.

Now we can obtain the short-time approximations for the solution by performing Laplace inversion[14]. We find that

$$\varphi\,(r,\theta,t) \propto \frac{H(t-t_1)}{2t_1\,(t-t_1)^{1/2}} + A_1\,B_1\,\frac{H(t-t_2)}{(t-t_2)^{1/2}} \quad (r_0 \geqslant r \geqslant a) \qquad \ldots(3.4)$$

$$\psi\,(r,\theta,t) \propto A_2\,B_2\,\frac{H(t-t_3)}{(t-t_3)^{1/2}}, \quad (r \geqslant a) \qquad \ldots(3.5)$$

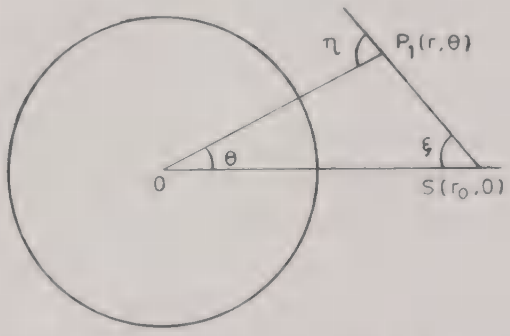

FIG. 2. Geometrical interpretation of the saddle point for the incident P pulse.

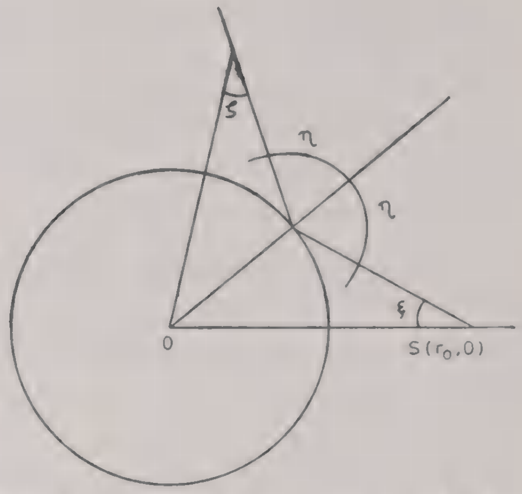

FIG. 3. Geometrical interpretation of the saddle point for the reflected P pulse.

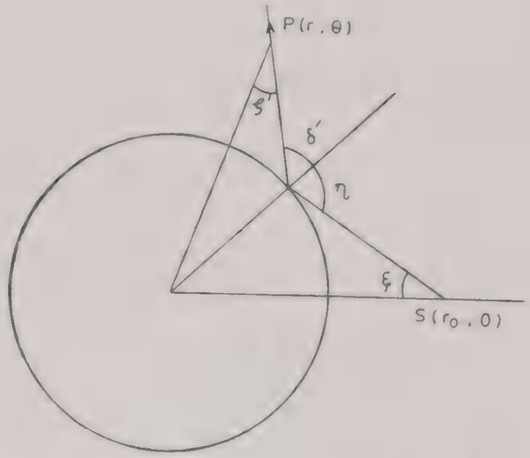

FIG. 4. Geometrical interpretation of the saddle point for the reflected S pulse.

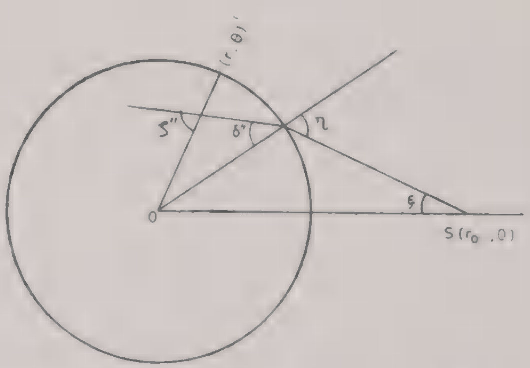

FIG. 5. Geometrical interpretation of the saddle point for the refracted P pulse.

$$\varphi_0 (r, \theta, t) \propto A_3 B_3 \frac{H(t - t_4)}{(t - t_4)^{1/2}}, \; (r \leqslant a) \qquad \text{...(3.6)}$$

Figures 6-8 present the numerical evaluation of these pulses respectively.

COMPARISON

Hwang *et al.*[13] discussed the problem of three dimensional elastic wave scattering due to a rigid cylinder in case of a compressional point source on comparing our results with those obtained by them, we find that in addition to the results obtained by us, they obtain an additional event in the illuminated region which they term as

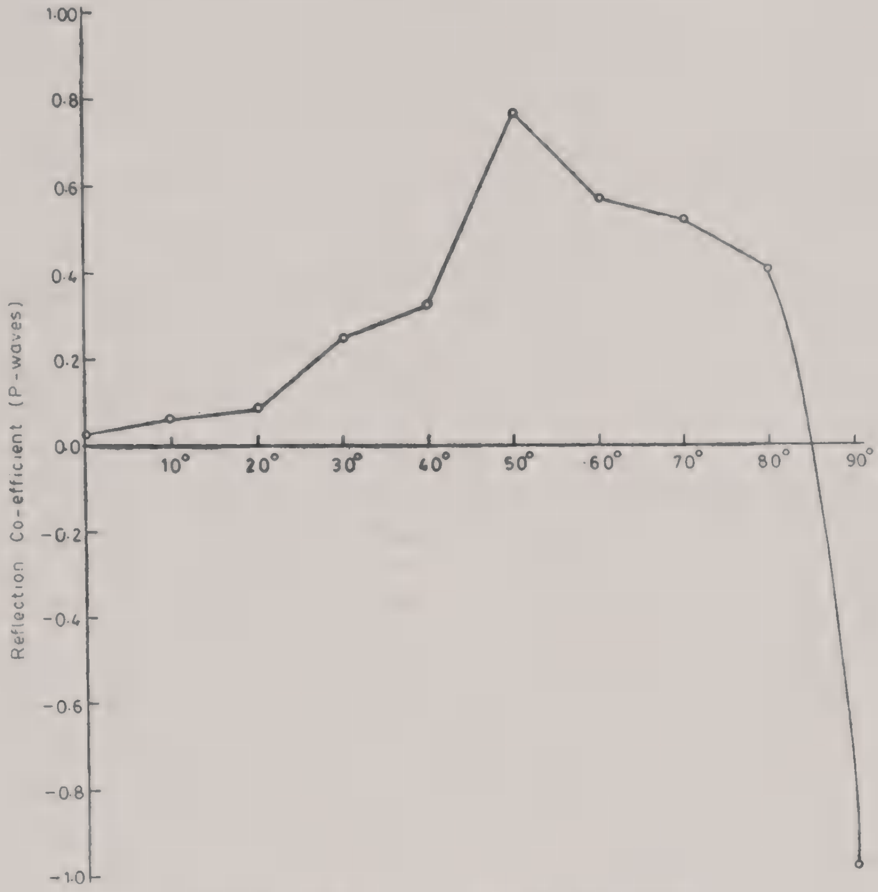

FIG. 6. Reflection co-efficients for P waves.

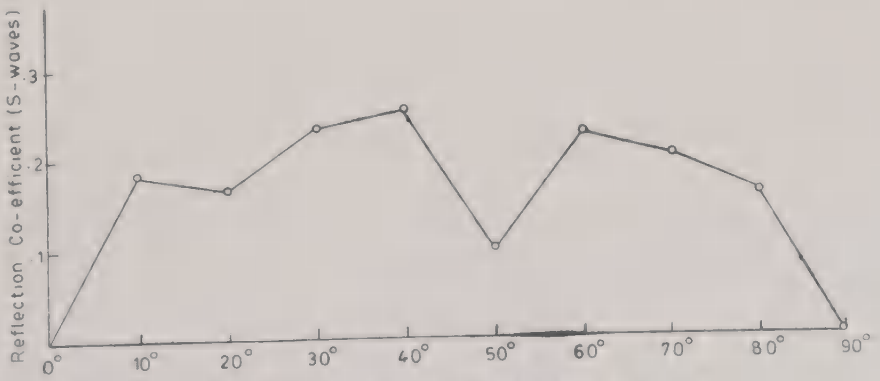

FIG. 7. Reflection co-efficients for S waves.

diffracted PP_dS wave. It may be pointed out that this possibility does not hold in the illuminated region. The physical ground for this is that diffracted waves exist in the shadow region only. Diffracted field is in fact the contribution of poles. It is obtained on evaluating the integrals (2.4) and (2.5) by Watsons residue method. Besides the results obtained by the Saddlepoint method are interpretable only in the

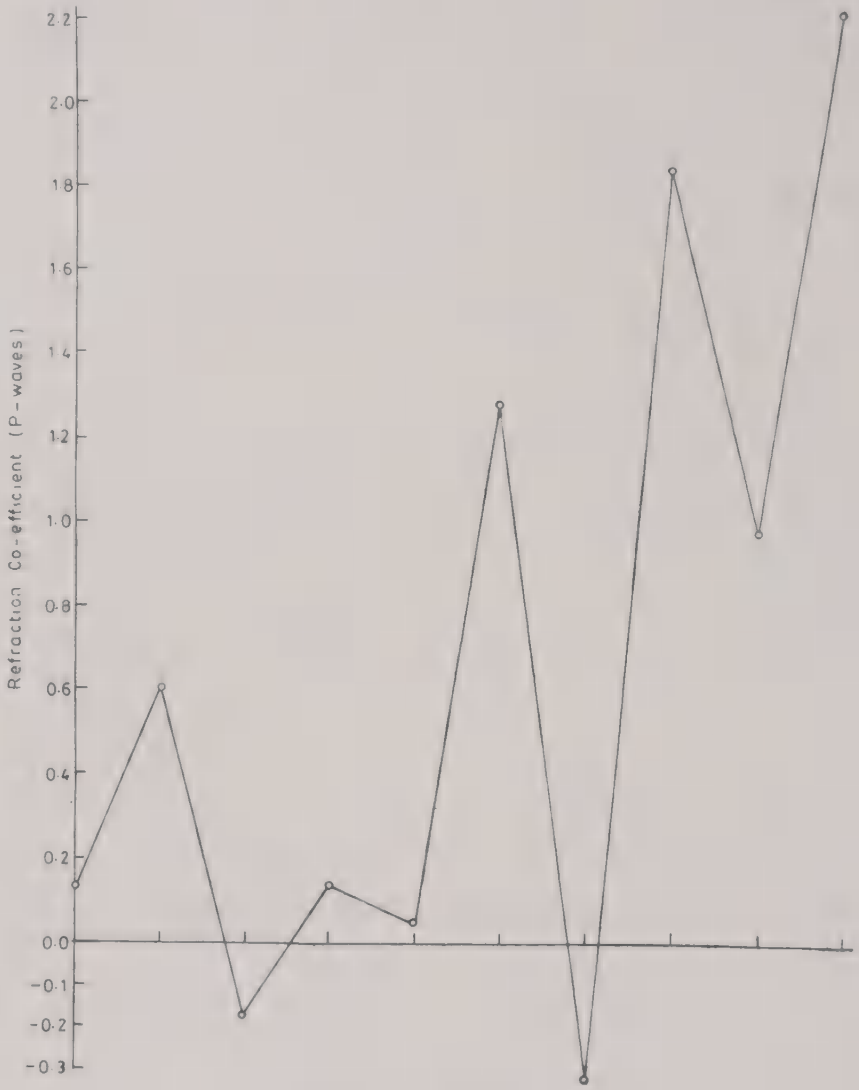

FIG. 8. Transmission co-efficient for P waves.

illuminated region[1]. This conclusion seems quite reasonable. It agrees with the results obtained by Mishra[15,16] and Rajhans and Agarwal[3].

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HALL EFFECTS ON THERMOSOLUTAL INSTABILITY OF A PLASMA

R. C. SHARMA AND NEELA RANI

Department of Mathematics, Himachal Pradesh University, Shimla 171005

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The thermosolutal instability of a plasma in the presence of a uniform horizontal magnetic field is studied to include the effects of Hall current. When the instability sets in as stationary convection, the Hall currents and the stable solute gradient are found to have destabilizing and stabilizing effects respectively. The case of overstability is also considered wherein the necessary conditions for the existence of overstability are derived.

1. INTRODUCTION

The theory of thermal convection of fluid layer heated from below under varying assumptions of hydromagnetics has been treated in detail by Chandrasekhar[1]. Gupta[2] has studied the problem of thermal instability in the presence of Hall current. The thermal instability of a compressible Hall plasma has been considered by Sharma[4]. Veronis[6] has studied the problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient, whereas the problem of thermohaline convection in a horizontal layer of viscous fluid heated from below and salted from above has been considered by Nield[3]. For thermohaline convection, buoyancy forces can arise not only from density differences due to variations in temperature, but also from those due to variations solute concentration.

The physics is quite similar in the stellar case in that helium acts like salt in raising the density and in diffusing more slowly than heat. The conditions under which convective motions are important in stellar atmospheres are usually far removed from consideration of single component fluid and rigid boundaries and therefore, it is desirable to consider a fluid acted on by a solute gradient and free boundaries. The problem of the onset of thermal instability in the presence of a solute gradient is of great importance because of its application to atmospheric physics and astrophysics, especially in the case of the ionosphere and the outer layers of the solar atmosphere. The Hall effects are likely to be important in these regions. A reconsideration of the thermosolutal instability including Hall effects is certainly called for and is the object of the present paper.

2. FORMULATION OF THE PROBLEM AND DISPERSION RELATION

Here we study the effect of Hall current on the thermosolutal instability of a plasma in the form of an infinite horizontal layer of thickness d, acted on by a horizontal

magnetic field $\vec{H}$ $(H, 0, 0)$ and a gravity force $\vec{g}$ $(0, 0, -g)$. This layer is heated from below and subjected to a stable solute gradient so that the temperatures and concentrations at the bottom surface $z = 0$ are T_0 and C_0 and at the upper surface $z = d$ and T_1 and C_1. Consider the cartesian coordinates (x, y, z) with the origin on the lower boundary $z = 0$ and the z-axis perpendicular to it along the vertical. The layer is heated and soluted from below such that a uniform temperature gradient β $(= | dT/dz |)$ and uniform solute gradient β' $(= | dC/dz |)$ are maintained.

Let $\vec{q}$ (u, v, w), p, ρ, T, C, α, α', g, η, N and e denote respectively the velocity, pressure, density, temperature, solute concentration, thermal coefficient of expansion, an analogous solvent coefficient of expansion, gravitational acceleration resistivity, number density and the charge of an electron. Then the equations expressing the conservation of momentum, mass, temperature, solute mass concentration and equation of state are

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\vec{q} + \vec{g}\left(1 + \frac{\delta\rho}{\rho_0}\right)$$

$$+ \frac{\mu_e}{4\pi\rho_0}(\nabla\times\vec{H})\times\vec{H} \qquad \text{...(1)}$$

$$\nabla.\vec{q} = 0 \qquad \text{..(2)}$$

$$\frac{\partial T}{\partial t} + (\vec{q}.\nabla)T = \kappa\nabla^2 T \qquad \text{...(3)}$$

$$\frac{\partial C}{\partial t} + (\vec{q}.\nabla)C = \kappa'\nabla^2 C \qquad \text{...(4)}$$

$$\rho = \rho_0[1 - \alpha(T - T_0) + \alpha'(C - C_0) \qquad \text{...(5)}$$

where the suffix zero refers to values at the reference level $z = 0$ and in writing eqn. (1), use has been made of the Boussinesq approximation. The magnetic permeability μ_e, the kinematic viscosity ν, the thermal diffusivity κ and the solute diffusivity κ' are all assumed to be constant.

Maxwell's equations give

$$\frac{\delta\vec{H}}{\delta t} = (\vec{H}.\nabla)\vec{q} + \eta\nabla^2\vec{H} - \frac{1}{4\pi Ne}\nabla\times[(\nabla\times\vec{H})\times\vec{H}] \qquad \text{...(6)}$$

$$\nabla.\vec{H} = 0 \qquad \text{...(7)}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{q}.\nabla$ stands for the convective derivative.

The steady state solution is

$$\vec{q} = 0, T = T_0 - \beta z, C = C_0 - \beta' z, \rho = \rho_0(1 + \alpha\beta z - \alpha'\beta' z). \qquad \text{...(8)}$$

Consider a small perturbation on the steady state solution and let $\vec{q}\,(u, v, w)$, δp, $\delta\rho$, θ, γ and $\vec{h}\,(h_x, h_y, h_z)$ denote respectively the perturbations in velocity, pressure, density, temperature, concentration and magnetic field. The change in density $\delta\rho$, caused by the perturbations θ and γ in temperature and concentration, is given by

$$\delta\rho = -\rho_0(\alpha\theta - \alpha'\gamma). \qquad ...(9)$$

Then the linearized hydromagnetic perturbation equations become

$$\frac{\partial\vec{q}}{\partial t} = -\nabla\delta p + \nu\nabla^2\vec{q} - \vec{g}(\alpha\theta - \alpha'\gamma) + \frac{\mu_e}{4\pi\rho_0}(\nabla\times\vec{h})\times\vec{H} \qquad ...(10)$$

$$\nabla\cdot\vec{q} = 0 \qquad ...(11)$$

$$\frac{\partial\vec{h}}{\partial t} = (\vec{H}.\nabla)\vec{q} + \eta\nabla^2\vec{h} - \frac{1}{4\pi Ne}\nabla\times[(\nabla\times\vec{h})\times\vec{H}] \qquad ...(12)$$

$$\nabla.\vec{h} = 0 \qquad ...(13)$$

$$\frac{\partial\theta}{\partial t} = \beta w + \kappa\nabla^2\theta \qquad ...(14)$$

$$\frac{\partial\gamma}{\partial t} = \beta' w + \kappa'\nabla^2\gamma. \qquad ...(15)$$

Analyzing the disturbances in terms of normal modes, we assume that the perturbation quantities are of the form

$$[w, \theta, \gamma, h_z, \zeta, \xi] = [W(z), \Theta(z), \Gamma(z), K(z), Z(z), X(z)]$$
$$\exp(ik_x x + ik_y y + nt) \qquad ...(16)$$

where n is the growth rate; k_x, k_y are wave numbers along the x and y directions and $k = \left(k_x^2 + k_y^2\right)^{1/2}$ is the resultant wave number $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ and $\xi = \frac{\partial h_y}{\partial x} - \frac{\partial hx}{\partial y}$ stand respectively for the z-components of vorticity and current density.

Letting $a = kd$, $\sigma = nd^2/\nu$, $p_1 = \nu/\kappa$, $p_2 = \nu/\eta$, $q = \nu/\kappa'$, $D = \frac{d}{dz}$ and putting the coordinates x, y, z in the new unit of length d, eqns. (10) – (15), using expression (16), in nondimensional form give

$$(D^2 - a^2)(D^2 - a^2 - \sigma)W - \left(\frac{ga^2d^2}{\nu}\right)(\alpha\Theta - \alpha'\Gamma) + ik_x\frac{\mu_e Hd^2}{4\pi\rho_0\nu}$$
$$\times(D^2 - a^2)K = 0 \qquad ...(17)$$

$$(D^2 - a^2 - \sigma)Z = -ik_x\frac{\mu_e Hd^2}{4\pi\rho_0\nu}X \qquad ...(18)$$

$$(D^2 - a^2 - p_2 \sigma) K = - ik_x \frac{Hd^2}{\eta} W + ik_x \frac{Hd^2}{4\pi Ne\eta} X \qquad \ldots(19)$$

$$(D^2 - a^2 - p_2 \sigma) X = - ik_x \frac{Hd^2}{\eta} Z - ik_x \frac{Hd^2}{4\pi Ne\eta} (D^2 - a^2) K \qquad \ldots(20)$$

$$(D^2 - a^2 - p_1\sigma) \Theta = - \frac{\beta d^2}{\kappa} W \qquad \ldots(21)$$

$$(D^2 - a^2 - q\sigma) \Gamma = - \frac{\beta' d^2}{\kappa} W. \qquad \ldots(22)$$

Eliminating Z, K, X, Θ and Γ between eqns. (17)—(22), we obtain

$$\begin{aligned}&(D^2 - a^2)(D^2 - a^2 - \sigma)^2 (D^2 - a^2 - p_2\sigma)^2 (D^2 - a^2 - p_1\sigma)(D^2 - a^2 \\ &- q\sigma) W + Q a^2 \cos^2\theta (D^2 - a^2)(D^2 - a^2 - p_1\sigma)(D^2 - a^2 - q\sigma) \\ &\{2(D^2 - a^2 - \sigma) . (D^2 - a^2 - p_2\sigma) + Q a^2 \cos^2\theta\} W - M a^2 \\ &\cos^2\theta (D^2 - a^2)^2 (D^2 - a^2 - \sigma)^2 (D^2 - a^2 - p_1\sigma)(D^2 - a^2 - q\sigma) \\ &W + \{Ra^2 (D^2 - a^2 - q\sigma) - Sa^2 (D^2 - a^2 - p_1\sigma)\} \\ &[(D^2 - a^2 - p_2\sigma)^2 (D^2 - a^2 - \sigma) + Q a^2 \cos^2\theta (D^2 - a^2 \\ &- p_2\sigma) - M a^2 \cos^2\theta (D^2 - a^2)(D^2 - a^2 - \sigma)] W = 0\end{aligned} \qquad \ldots(23)$$

where $M = (H/4\pi Ne\eta)^2$ is nondimensional number accounting for the Hall currents, $Q = \mu_e H^2 d^2/4\pi\rho_0\nu\eta$ is the Chandrasekhar number, $R = g\alpha\beta d^4/\nu\kappa$, is the thermal Rayleigh number, $S = \frac{g\alpha'\beta' d^4}{\nu\kappa'}$ is the analogous solute Rayleigh number and $\cos\theta = k_x/k$.

We consider the case of a fluid layer with two free surfaces when the adjoining medium is assumed to be electrically nonconducting. The case of two free boundaries is the most appropriate for stellar atmospheres[5]. The boundary conditions appropriate to the problem, using (16), are

$$W = D^2W = X = DZ = \Theta = \Gamma = 0 \qquad \ldots(24)$$

and the components of $\vec{h}$ are continuous. Because the components of the magnetic field are continuous and the tangential components are zero outside the fluid then

$$DK = 0 \qquad \ldots(25)$$

on the boundaries.

Using the boundary conditions (24) and (25), it can be shown with the help of eqns. (17) — (22) that all the even order derivatives of W vanish at the boundaries. Hence the proper solution of eqn. (23) characterizing the lowest mode is

$$W = W_0 \sin \pi z \qquad \ldots(26)$$

where W_0 is a constant.

Substituting (26) in eqn. (23) and letting $x = a^2/\pi^2$, $i\sigma_1 = \sigma/\pi^2$, $R_1 = R/\pi^4$, $S_1 = S/\pi^4$ and $Q_1 = Q/\pi^2$, we obtain the dispersion relation

$$R_1 = \frac{(1 + x)(1 + x + ip_1 \sigma_1)[\{(1 + x + i\sigma_1)(1 + x + ip_2\sigma_1) + Q_1 x \cos^2\theta\}^2 + Mx \cos^2\theta (1 + x)(1 + x + i\sigma_1)^2]}{\kappa [(1 + x + ip_2 \sigma_1)^2 (1 + x + i\sigma_1) + Q_1 x \cos^2\theta (1 + x + ip_2 \sigma_1) + Mx \cos^2\theta (1 + x)(Hx + i\sigma_1)]} + \frac{S_1 (1 + x + ip_1 \sigma_1)}{(1 + x + iq\sigma_1)}. \qquad \ldots(27)$$

3. The Stationary Convection

For the stationary convection, $\sigma = 0$ and eqn. (27) reduces to

$$R_1 = S_1 + \left(\frac{1 + x}{x}\right) \frac{\{(1 + x)^2 + Q_1 x \cos^2\theta\}^2 + Mx \cos^2\theta (1 + x)^3}{(1 + x)^2 + Q_1 x \cos^2\theta + Mx \cos^2\theta (1 + x)}. \qquad \ldots(28)$$

Equation (28) expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters S_1, Q_1 and M. To investigate the effect of stable solute gradient and Hall currents, we examine the nature of dR_1/dS_1 and dR_1/dM analytically. Equation (28) yields

$$\frac{dR_1}{dS_1} = +1 \qquad \ldots(29)$$

which implies that the stable solute gradient has a stabilizing effect on the thermosolutal convection. Equation (28) also yields

$$\frac{dR_1}{dM} = -\left(\frac{1 + x}{x}\right) \frac{Q_1 x^2 (1 + x) \cos^4\theta [(1 + x)^2 + Q_1 x \cos^2\theta]}{[(1 + x)^2 + Q_1 x \cos^2\theta + Mx (1 + x) \cos^2\theta]^2} \qquad \ldots(30)$$

which is negative. The Hall currents, therefore, have a destabilizing effect on the thermosolutal instability of a plasma.

4. The Overstable Case

Here we discuss the possibility as to whether instability may occur as overstability. Since for overstability we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it will suffice to find conditions for which (27) will admit of solutions with σ_1 real. Equating real and imaginary parts of eqn. (27) and eliminating R_1 between them, we obtain

$$A_4 c^4 + A_3 c^3 + A_2 c^2 + A_1 c + A_0 = 0 \qquad \ldots(31)$$

where we have put $c = \sigma_1^2$, $b = 1 + x$ and

$$A_4 = p_2^4 q^2 (1 + p_1)\, b \qquad \ldots(32)$$

$$\begin{aligned} A_0 = {} & (1 + p_1)\, b^9 + 2M (1 + p_1)(b - 1)\, b^8 \cos^2\theta \\ & + b^7 \{Q_1 \cos^2\theta\, (b - 1)(3p_1 - p_2 + 2) + M^2 \cos^4\theta\, (1 + p_1) \\ & \times (b - 1)^2\} + b^6 \{S_1 (p_1 - q)(b - 1) + MQ_1 \cos^4\theta\, (3p_1 \\ & + p_2 + 2)(b - 1)^2\} + b^5 \{Q_1^2 \cos^4\theta\, (3p_1 - 2p_2 + 1)(b-1)^2 \\ & + 2M S_1 \cos^2\theta\, (p_1 - q)(b - 1)^2\} + b^4 \{2Q_1 S_1 \cos^2\theta \\ & (p_1 - p_2)(b - 1)^2 + M^2 S_1 \cos^4\theta\, (p_1 - q)(b - 1)^2 \\ & + Q_1^2\ M \cos^6\theta\, (p_1 - 1)(b - 1)^3\} \\ & + b^3 \{Q_1^3 \cos^6\theta\, (p_1 - p_2)(b - 1)^3 + 2M S_1 Q_1 \cos^4\theta \\ & (p_1 - q)(b - 1)^3\} + Q_1^2 S_1 \cos^4\theta\, (p_1 - q)\, b^2 (b - 1)^3. \qquad \ldots(33) \end{aligned}$$

Since σ_1 is real overstability, the four values of c are positive. The product of the roots of (31) is A_0/A_4, which is positive if $A_0 > 0$ (since from (32), $A_4 > 0$), Equation (33) shows that A_0 is always positive if

$$p_1 > 1,\ p_1 > q \text{ and } p_1 > p_2 \qquad \ldots\ldots(34)$$

which means

$$\kappa < \nu,\ \kappa < \kappa' \text{ and } \kappa < \eta \qquad \ldots(35)$$

Thus $\kappa < \nu$, $\kappa < \kappa'$ and $\kappa < \eta$ are the necessary conditions for the existence of overstability

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THREE-DIMENSIONAL MAGNETOFLUIDDYNAMIC FLOW WITH PRESSURE GRADIENT AND FLUID INJECTION

M. G. TIMOL

Department of Mathematics, B. K. M. Science College, Valsad 396001, *Gujarat*

AND

A. G. TIMOL

I. P. Mission Higher Secondary Boys School, Muglisara, Surat 395003

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The non-linear partial differential equations of motion for three-dimensional flow of an incompressible viscous fluid flowing over a semi-infinite plate under the influence of a magnetic field and a pressure gradient, and with or without suction or injection through the plate wall are developed with necessary boundary conditions. The group-theoretic methods are employed to obtain proper similarity transformations. Similarity solution of such flow system yields coupled non-linear ordinary differential equations.

1. INTRODUTION

Due to increasing number of technical applications using magnetohydrodynamic effect, it is desirable to extend many of the available viscous hydromagnetic solutions to include the effects of magnetic fields for those cases when the viscous fluid is electrically conducting. Flow past a flat plate has been studied in Rossow[1] and Carrier and Greenspan[2]. Heat transfer for these cases has been discussed in Rossow[1] and Afzal[3]. Rossow[1] has considered transverse magnetic field whereas Carrier and Greenspan[2] have studied the effect of a longitudinal magnetic field on the velocity and the temperature distributions. There have been other computations of viscous magnetohydrodynamic flow reported in the literature[4,5]. Djukic[4] has considered only Hiemenz magnetic flow of non-Newtonian power-law fluid. He has solved the governing non-linear ordinary differential equations using Galerkin's technique, whereas Srivastava and Usha[5] has discussed magnetofluiddynamics of simple two-dimensional flow with pressure gradient and fluid injection. Recently such case is extended by Timol[6] for a non-Newtonian power-law fluid.

2. BASIC EQUATIONS

The basic equations of magnetofluiddynamics and coventional fluiddynamics are different by only additional force term due to electromagnetic field in momentum equation and a term due to Joule heating in the energy equation. In such situation

the Maxwell's equations have to be satisfied in the entire flow field as well as in the body and interface.

In order to derive the basic equations, the following assumptions are made :

(1) The fluid under consideration is incompressible finitely conducting with constant physical properties.

(2) Hall effect, electrical and polarization effects are neglected.

(3) The induced magnetic field is neglected.

(4) The flow is steady and laminar and the imposed magnetic field is perpendicular to the free stream velocities.

(5) The magnetic Reynolds number is assumed to be small.

Under these assumptions we now write continuity and momentum equations governing the velocity distribution in the presence of magnetic field as,

$$\nabla \cdot \bar{V} = 0 \qquad \ldots(1)$$

$$\rho \bar{V} \cdot \nabla \bar{V} = - \nabla P + \rho \gamma_H \nabla^2 \bar{V} + \bar{J} \times \bar{B} \qquad \ldots(2)$$

where the third term on the right-hand side of eqn. (2) is the Lorentz force due to the magnetic $\bar{B}$, and is given by

$$\bar{J} \times \bar{B} = \sigma (\bar{V} \times \bar{B}) \times \bar{B}. \qquad \ldots(3)$$

We now consider the flow past a semi-infinite flat plate placed in the direction of the flow. The plate is in $X - Z$ plane and is between $0 \leqslant x < \infty$ and $-\infty < Z < \infty$ and free stream is in the X-direction as shown in Fig. 1. Here we shall consider that all flow quantities are independent of the Z-coordinate. Such flows are characterized by the fact that their stream lines form a system of 'translates'. That is stream line pattern can be obtained by translating any particular stream line parallel to the leading edge of the surface[7]. It is hoped that by assuming independence of flow quantities in one direction, more quantitative information may be obtained on the characteristics of three-dimensional boundary layer flows.

Thus the problem considered here is essentially a quasi-two-dimensional one. The magnetic field vector $\bar{B}$ is perpendicular to the free stream and is along Y-direction because the induced magnetic field is neglected. The basic equations of the steady, viscous, laminar, there-dimensional boundary layer flow of a fluid flowing over a semi-infinite flat plate under the influence of transverse magnetic field and pressure gradient with or without suction or injection through the plate wall can be derived from the eqns. (1) and (2) on the basis of usual boundary layer assumptions[8], as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \ldots(4)$$

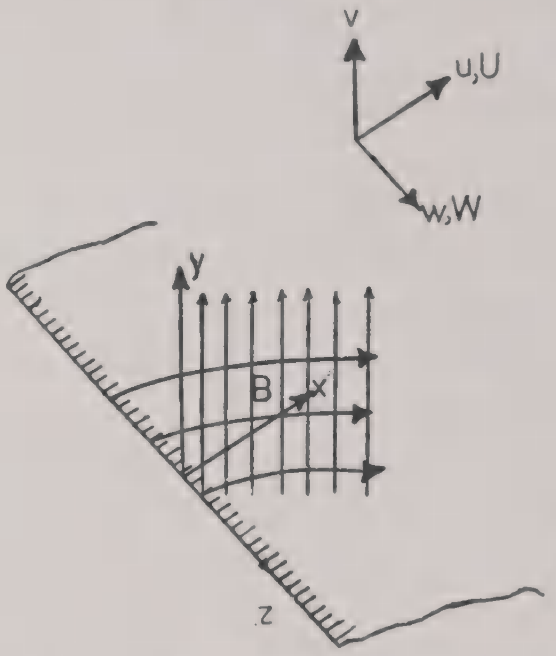

FIG. 1. Flow over a plate in rectangular coordinate system under the influence of transverse magnetic field.

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{g}{\rho} \frac{dp}{dx} + \gamma_H \frac{\partial^2 u}{\partial y^2} - [g \sigma B_y^2 (x)/\rho] u \quad \ldots(5)$$

$$u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} = - \frac{g}{\rho} \frac{dp}{dz} + \gamma_H \frac{\partial^2 w}{\partial y^2} - [g \sigma B_y^2 (x)/\rho] w. \quad \ldots(6)$$

The boundary conditions for such flow system are taken as,

Case I

$$y = 0 \Rightarrow u = 0;\ v = V_0\, x^{(m-1)/2};\ w = 0 \quad \ldots(7)$$

$$y \to \infty \Rightarrow u \to U_0\, x^m;\ w \to W(x) = W\, x^m. \quad \ldots(8)$$

Case II

$$y = 0 \Rightarrow u = 0;\ v = V_0 \exp\left(\frac{mx}{2}\right);\ w = 0 \quad \ldots(9)$$

$$y \to \infty \Rightarrow u \to U(x) = U_0 \exp(mx)$$

$$w \to W(x) = W_0 \exp(mx) \quad \ldots(10)$$

where x and z are the coordinates parallel to the plate, y a distance from the plate; u, v, w the velocity components of the fluid along x, y and z directions respectively; U, W velocity components in the main flow along x and z direction respectively; g the acceleration of the gravity; P the pressure; γ_H the magnetic viscosity of the fluid $(= m/\rho)$ ρ—the fluid density; m a physical constant, σ (Greek 'sigma') the electrical

conductivity of the fluid; B_Y the imposed magnetic induction parallel to y axis; U_0, V_0, W_0, $m > 0$ are constants.

Now introduce stream function $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x}. \qquad \ldots(11)$$

Then, the equation of continuity (4) gets satisfied identically and equations of motion (5) and (6) with boundary conditions (7)-(10) will be,

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \, \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = G_1(x) + \gamma_H \frac{\partial^3 \psi}{\partial y^3} - H_1(x)\frac{\partial \psi}{\partial y} \qquad \ldots(12)$$

$$\frac{\partial \psi}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial w}{\partial y} = G_2(x) + \gamma_H \frac{\partial^2 w}{\partial y^2} - H_1(x)\, w \qquad \ldots(13)$$

where

$$G_1(x) = -\, g/\rho \, \frac{dp}{dx} \qquad \ldots(14)$$

$$G_2(x) = -\, g/\rho \, \frac{dp}{dz} \qquad \ldots(15)$$

$$H_1(x) = \frac{g\sigma \, B_y^2(x)}{\rho}. \qquad \ldots(16)$$

Then the boundary conditions are,

Case I

$$y = 0 \Rightarrow \frac{\partial \psi}{\partial y} = 0; \; \frac{\partial \psi}{\partial x} = -\, V_0 \, x^{(m-1)/2}; \; w = 0 \qquad \ldots(17)$$

$$y \to \infty \Rightarrow \frac{\partial \psi}{\partial y} \to U(x) = U_0 \, x^m; \; w \to W(x) = W_0 \, x^m. \qquad \ldots(18)$$

Case II

$$y = 0 \Rightarrow \frac{\partial \psi}{\partial y} = 0, \; \frac{\partial \psi}{\partial x} = -\, V_0 \exp\left(\frac{m-1}{2}\right); \; w = 0 \qquad \ldots(19)$$

$$y \to \infty \Rightarrow \frac{\partial \psi}{\partial y} \to U(x) = U_0 \exp(mx); \; w \to W(x) = W_0 \exp(mx). \qquad \ldots(20)$$

The goal of reducing partial differential equations (12) and (13) to ordinary differential equations with the meaningful boundary conditions (17) -(20) by similarity technique will cause restrictions to be imposed on the functions $G_1(x)$, $G_2(x)$ and $H_1(x)$. In order to obtain suitable similarity variables, the group theoretic method is employed.

3. Group-Theoretic Analysis

Similarity analysis by the group-theoretic technique is based on the concept derived from the theory of transformations of groups. Recently this technique is found to give adequate treatment of boundary layer equations. The basic concept of this method was first introduced by Birkhoff[9] and later on it was extended exclusively by Hansen[10], Morgan[11], Ames[12] and others. For the present problem we shall employ two distinct classes of one parameter groups of transformation—a linear group of transformation and a spiral group of transformation and also we shall discuss both the cases separately.

Case 1

A one parameter group of transformation Γ_1 is selected as,

$$\Gamma_1 = \begin{cases} x = A^{\alpha_1}\,\bar{x} & y = A^{\alpha_2}\,\bar{y} \\ \psi = A^{\alpha_3}\,\psi & w = A^{\alpha_4}\,\bar{w} \end{cases}$$

where α_1, α_2, α_3, α_4 and A are real arbitrary constants.

We now seek relations among α's such that the basic equations (12), (13) along with the boundary conditions (17) and (18) will be invariant under this group of transformation Γ_1. This suggests that $G_1(x)$, $G_2(x)$ and $H_1(x)$ are to be selected as

$$G_1(x) = g_0\, x^{\alpha_5} \quad \text{...(21)}$$

$$H_1(x) = h_0\, x^{\alpha_6} \quad \text{...(22)}$$

$$G_2(x) = g_1\, x^{\alpha_7}. \quad \text{...(23)}$$

Now invariant conditions demand that the power of A in each term of the transformed equations should be equal, which yields

$$2\alpha_3 - 2\alpha_2 - \alpha_1 = \alpha_5\,\alpha_1 = \alpha_3 - 3\,\alpha_2 = \alpha_6\,\alpha_1 + \alpha_3 - \alpha_2 \quad \text{...(24)}$$

$$\alpha_4 + \alpha_3 - \alpha_2 - \alpha_1 = \alpha_7\,\alpha_1 = \alpha_4 - 2\alpha_2 = \alpha_6\,\alpha_1 + \alpha_4. \quad \text{...(25)}$$

Further, from the boundary conditions (17), (18), we get,

$$\alpha_3/\alpha_1 = \frac{m+1}{2};\ \alpha_2/\alpha_1 = \frac{1-m}{2};\ \frac{\alpha_4}{\alpha_1} = m \quad \text{...(26)}$$

and from (24), (25) and (26), we get,

$$\alpha_5 = 2m - 1;\ \alpha_6 = m - 1;\ \alpha_7 = 2m - 1. \quad \text{...(27)}$$

Now invariants of the group Γ_1 are,

$$ay/x^{\alpha_2/\alpha_1};\ \psi/bx^{\alpha_3/\alpha_1} \text{ and } w/cx^{\alpha_4/\alpha_1}$$

where a, b and c are constants. This enable us to derive the following similarity independent and dependent variables :

$$\eta = ay/x^{(m-1)/2} \qquad ...(28)$$

$$f_1(\eta) = \psi/b\, x^{(m+1)/2} \qquad ...(29)$$

$$f_2(\eta) = w/c\, x^m. \qquad ...(30)$$

Substituting eqns. (28), (29) and (30) in eqns. (12) and (1:) and simplifying, we get,

$$\lambda_1 f_1'^2 - f_1'' f_1 = \frac{\gamma_H a^2}{\lambda_2 U_0} f_1''' + \frac{g_0}{\lambda_2 U_0^2} - \frac{h_0}{\lambda_2 U_0} f_1' \qquad ...(31)$$

$$\lambda_1 f_1' f_2 - f_1 f_2' = \frac{Ha^2}{\lambda_2 U_0} f_2'' + \frac{g_1}{\lambda_2 U_0 W_0} - \frac{h_0}{\lambda_2 U_0} f_2. \qquad ...(32)$$

where

$$\lambda_1 = \frac{2m}{m+1};\ \lambda_2 = \frac{m+1}{2} \qquad ...(33)$$

$$G_1(x) = g_0\, x^{2m-1} \qquad ...(34)$$

$$G_2(x) = g_1\, x^{2m-1} \qquad ...(35)$$

$$H_1(x) = h_0\, x^{m-1} \qquad ...(36)$$

and a prime denotes differentiation with respect to η.

Also from the boundary condition (18), it follows that

$$U_0 = ab;\ W_0 = c. \qquad ...(37)$$

Setting

$$S_0 = \frac{g_0}{\lambda_2 U_0^2},\ S_1 = \frac{g_1}{\lambda_2 U_0 W_0} \text{ and } \mu_0 = \frac{h_0}{\lambda_2 U_0} \qquad ...(38)$$

and

$$\frac{\gamma_H a^2}{\lambda_2 U_0} = 1 \qquad ...(39)$$

eqns. (31) and (32) will become

$$f_1''' + f_1 f_1'' - \lambda_1 f_1'^2 + S_0 - \mu_0 f_1' = 0 \qquad ...(40)$$

$$f_2'' + f_1 f_2' - \lambda_1 f_1' f_2 + S_1 - \mu_0 f_2 = 0. \qquad ...(41)$$

Now when free stream velocites are function of x, then pressure gradients will be given by :

$$-(g/\rho)\frac{dp}{dx} = U\frac{dU}{dx} + \frac{g\,\sigma\,B_y^2(x)}{\rho} U + \partial U/\partial t \qquad ...(42)$$

$$-(g/\rho)\frac{dp}{dz} = U\frac{dw}{dx} + \frac{g\sigma B_y^2(x)}{\rho} W + \partial W/\partial t. \quad \text{...(43)}$$

But for steady flow, free stream velocities outside the boundary layer is independent of time i.e.

$$\frac{\partial U}{\partial t} = 0; \frac{\partial W}{\partial t} = 0 \quad \text{...(44)}$$

under (14)—(16), (17) and (34)—(35), the relationship of (42)—(43) will be,

$$S_0 = \lambda_1 + \mu_0 \quad \text{...(45)}$$

$$S_1 = \lambda_1 + \mu_0. \quad \text{...(46)}$$

Substituting (45) and (46) in the equations (40) and (41) respectively, we get,

$$f_1''' + f_1 f_1'' + \lambda_1 (1 - f_1'^2) + \mu_0 (1 - f_1') = 0 \quad \text{...(47)}$$

$$f_2'' + f_1 f_2' + \lambda_1 (1 - f_1' f_2) + \mu_0 (1 - f_2) = 0. \quad \text{...(48)}$$

Equations (47), (48) are coupled non-linear ordinary differential equations for the function of $f(\eta)$. With the transformed boundary conditions,

$$\eta = 0 \Rightarrow f_1' = \frac{-V_0}{b\lambda_2}, f_2 = 0 \quad \text{...(49)}$$

$$\eta \to \infty \Rightarrow f_1' = 1, f_2 = 1. \quad \text{...(50)}$$

From equations (37) and (39) values of a, b and c will be

$$a = \left(\frac{\lambda_2 U_0}{\gamma_H}\right)^{1/2}; b = \left(\frac{\gamma_H U_0}{\lambda_2}\right)^{1/2}; c = W_0. \quad \text{...(51)}$$

Case II

One parameter spiral group of transformation Γ_2 can be chosen in the form,

$$\Gamma_2 = \begin{cases} x = \beta_1 b + \bar{x}, \ y = e^{\beta_2 b}\bar{y} \\ \psi = e^{\beta_3 b}\bar{\psi} \quad w = e^{\beta_4 b}\bar{w} \end{cases}$$

where β_1, β_2, β_3, β_4 and b are real constants and e is a real parameter of the group transformation Γ_2.

In this case Γ_2 enables us that $G_1(x)$, $G_2(x)$ and $H_1(x)$ are to be selected in such a way,

$$G_1(x) = g_0 e^{\beta_5 x} \quad \text{...(52)}$$

$$G_2(x) = g_1 e^{\beta_6 x} \quad \text{...(53)}$$

$$H_1(x) = h_0 e^{\beta_7 x} \quad \text{...(54)}$$

following the same procedure as in case-I, the followining absolute invariants are obtained :

$$\xi = a_1 Y/\exp(\tfrac{1}{2} mx) \quad \text{...(55)}$$

$$\psi = b_1 \exp(mx) F_1(\xi) \quad \text{...(56)}$$

$$w = c_1 \exp(mx) F_2(\xi) \quad \text{...(57)}$$

where

$$a_1 = \left(\frac{mU_0}{2\gamma_H}\right)^{1/2} b_1 = \left(\frac{2\gamma_H U_0}{m}\right)^{1/2} c_1 = W_0. \quad \text{...(58)}$$

Under (55)—(57), finally eqns. (12)—(13) with the boundary conditions (18), (19) will be transformed to following non-linear ordinary differential equation :

$$F_1''' + F_1 F_1'' + 2(1 - F_1'^2) + \mu_0 (1 - F_1') = 0 \quad \text{...(59)}$$

$$F_2'' + F_1 F_2' + 2(1 - F_1' F_2) + \mu_0 (1 - F_3) = 0 \quad \text{...(60)}$$

where $\mu_0 = \dfrac{2h_0}{mU_0}$ and prime denotes differentiation with respect to ξ

with the boundary conditions

$$\xi = 0; \; F_1' = \frac{-2V_0}{(m\gamma_H U_0)^{1/2}}; \; F_2 = 0 \quad \text{...(61)}$$

$$\xi = \infty; \; F_1' = 1; \; F_2 = 1. \quad \text{...(62)}$$

Equations (59), (60) with the boundary conditions (61), (62) constitute a pair of non-linear ordinary coupled differential equations.

4. Conclusion

The analysis of laminar, incompressible, three-dimensional magnetofluiddynamic boundary layer equations of viscous fluids with stream lines forming a system of 'translates' lead to the following conclusions :

(i) If we take $V_0 = 0$ in eqns. (49) and (61), we have magnetofluiddynamic boundary layer problem without suction or injection through the plate wall which either in Case-I or in Case-II is reduced to a solution of a boundary value problem of following third-order non-linear coupled ordinary differential equations :

$$f_1''' + f_1 f_1'' + \lambda_1 (1 - f_1'^2) + \mu_0 (1 - f_1') = 0 \quad \text{...(63)}$$

$$f_2'' + f_1 f_2' + \lambda_1 (1 - f_1' f_2) + \mu_0 (1 - f_2) = 0 \quad \text{...(64)}$$

where

$$\lambda_1 > 0; \mu_0 > 0.$$

The boundary conditions are

$$f_1(0) = 0; f_1'(0) = 0; f_2(0) = 0 \quad ...(65)$$

$$f_1'(\infty) = 1, f_2(\infty) = 1. \quad ...(66)$$

(ii) For the non-magnetic case i.e. for $\mu_0 = 0$, the above set of eqns. (63)—(64) will reduce to the set of equations for simple three-dimensional viscous incompressible flows obtained by Timol *et al.*[13] (for power index $n = 1$ and $\lambda_1 = 1/3$).

(iii) For $m = 1$, $U(x) = U_0 x$, $W(x) = W_0 x$, the entire problem will reduce to the particular case of three-dimensional magnetic flow near a stagnation point i.e. Hiemenz three-dimensional magnetic flows. In such situation the transverse magnetic field will be found constant through the plate wall. The present problem is recently extended to include non-Newtonian fluids of different models by Timol[14].

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